

HIGH ORDER APPROXIMATION FOR THE BOLTZMANN EQUATION WITHOUT ANGULAR CUTOFF

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ABSTRACT. In order to solve the Boltzmann equation numerically, in the present work, we propose a new model equation to approximate the Boltzmann equation without angular cutoff. Here the approximate equation incorporates Boltzmann collision operator with angular cut-off and the Landau collision operator. As a first step, we prove the well-posedness theory for our approximate equation. Then in the next step we show the error estimate between the solutions to the approximate equation and the original equation. Compared to the standard angular cut-off approximation method, our method results in higher order of accuracy.

Keywords: homogeneous Boltzmann equation, long-range interactions, hard potentials, high order approximation.

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1. INTRODUCTION

1.1. The Boltzmann equation. Our interest is to consider the numerical method for the spatially homogeneous Boltzmann equation with long-range interaction in the case of hard potentials. Here, the spatial homogeneity means the unknown function is assumed to be independent of the position variables. In this case, the Boltzmann equation reads:

$$(1.1) \quad \partial_t f = Q(f, f),$$

where $f(t, v) \geq 0$ is the distribution function of collision particles which at time $t \geq 0$ move with velocity $v \in \mathbb{R}^3$. The Boltzmann collision operator Q is a bilinear operator which acts only on the velocity variables v , that is,

$$Q(g, h)(v) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} \int_{SS^2} B(v - v_*, \sigma) (g'_* h' - g_* h) d\sigma dv_*.$$

Here we use the standard shorthand $h = h(v)$, $g_* = g(v_*)$, $h' = h(v')$, $g'_* = g(v'_*)$ where v' , v'_* are given by

$$(1.2) \quad v' = \frac{v + v_*}{2} + \frac{|v - v_*|}{2} \sigma, \quad v'_* = \frac{v + v_*}{2} - \frac{|v - v_*|}{2} \sigma, \quad \sigma \in SS^2.$$

The nonnegative function $B(v - v_*, \sigma)$ in the collision operator is called the Boltzmann collision kernel. It is always assumed to depend only on $|v - v_*|$ and $\langle \frac{v - v_*}{|v - v_*|}, \sigma \rangle$. We introduce the angle variable θ through $\cos \theta = \langle \frac{v - v_*}{|v - v_*|}, \sigma \rangle$. Without loss of generality, we may assume that $B(v - v_*, \sigma)$ is supported in the set $0 \leq \theta \leq \frac{\pi}{2}$, i.e., $\langle \frac{v - v_*}{|v - v_*|}, \sigma \rangle \geq 0$, for otherwise B can be replaced by its symmetrized form:

$$\bar{B}(v - v_*, \sigma) = [B(v - v_*, \sigma) + B(v - v_*, -\sigma)] \mathbf{1}_{\langle \frac{v - v_*}{|v - v_*|}, \sigma \rangle \geq 0}.$$

Here, $\mathbf{1}_E$ is the characteristic function of the set E .

1.2. Assumptions on the collision kernel. We consider the collision kernel satisfying the following assumptions:

- (A-1) The cross-section $B(v - v_*, \sigma)$ takes a product form of

$$B(v - v_*, \sigma) = \Phi(|v - v_*|) b(\cos \theta),$$

where both Φ and b are nonnegative functions.

- (A-2) The angular function $b(t)$ is not locally integrable and it satisfies

$$K^{-1} \theta^{-1-2s} \leq \sin \theta b(\cos \theta) \leq K \theta^{-1-2s}, \quad \text{with } 0 < s < 1, \quad K \geq 1.$$

- (A-3) The kinetic factor Φ takes the form of

$$\Phi(|v - v_*|) = |v - v_*|^\gamma.$$

- (A-4) The parameter γ verifies that $0 < \gamma \leq 2$.

We remark that under assumption (A-2), we have $A_2 \stackrel{\text{def}}{=} \int_{SS^2} b(\cos \theta) \sin^2 \theta d\sigma < \infty$.

The solutions of the Boltzmann equation (1.1) have the fundamental physical properties of conserving the total mass, momentum and kinetic energy, that is, for all $t \geq 0$,

$$\int_{\mathbb{R}^3} f(t, v) \phi(v) dv = \int_{\mathbb{R}^3} f(0, v) \phi(v) dv, \quad \phi(v) = 1, v, |v|^2.$$

Moreover, there exists a quantity called entropy satisfying the Boltzmann's H theorem, which formally is

$$-\frac{d}{dt} \int_{\mathbb{R}^3} f \log f dv = - \int_{\mathbb{R}^3} Q(f, f) \log f dv \geq 0.$$

1.3. Existing results, motivations and difficulties. The well-posedness of the spatially homogeneous Boltzmann equation with angular cut-off, that is when $\int_0^{\pi/2} \sin \theta b(\cos \theta) d\theta < \infty$, had been investigated by many authors. For the hard potentials, Arkeryd [8] and Mischler-Wennberg [20] established the existence and uniqueness of the solutions in weighted L^1 space. Recently, Lu-Mouhot in [14] extended the results to the space of non-negative measure with finite non-increasing kinetic energy. For the well-posedness of the spatially homogeneous Boltzmann equation without angular cut-off, we refer to [10] and the references therein. As for the regularity theory of the equation, we refer to [18] for the analysis of the positive part of the collision operator and the propagation of smoothness in the case of angular cut-off and refer to [2], [4], [13] and [19] in the case of long-range interaction.

For any $0 < \epsilon \leq \frac{\sqrt{2}}{2}$, let $b^\epsilon = b \mathbf{1}_{\sin \frac{\theta}{2} \geq \epsilon}$, and Q^ϵ be the operator associated to the angular cut-off kernel $B^\epsilon(v - v_*, \sigma) = |v - v_*|^\gamma b^\epsilon(\cos \theta)$. That is,

$$Q^\epsilon(g, h)(v) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} \int_{SS^2} B^\epsilon(v - v_*, \sigma) (g'_* h' - g_* h) d\sigma dv_*.$$

Then the angular cut-off Boltzmann equation

$$(1.3) \quad \partial_t f = Q^\epsilon(f, f)$$

is well-posed(see [12]). And moreover if f and f^ϵ are solutions to the Boltzmann equation (1.1) and the cutoff Boltzmann equation (1.3) with the same initial datum f_0 respectively, then one has

$$f = f^\epsilon + O(\epsilon^{2-2s}).$$

The cut-off Boltzmann operator Q^ϵ omits all grazing collisions and then results in an error of order $2 - 2s$. We emphasize that the cutoff Boltzmann equation (1.3) is not a good approximation to the Boltzmann equation (1.1) as the singularity parameter s approaches to 1.

The effect of grazing collisions has been studied extensively, and we refer to [5] and [9]. It is proved that the limit of concentrating grazing collisions leads to the Landau collision operator. Mathematically, if denote $b_\epsilon = b \mathbf{1}_{\sin \frac{\theta}{2} \leq \epsilon}$, and let Q_ϵ be the operator associated to $B_\epsilon(v - v_*, \sigma) = |v - v_*|^\gamma b_\epsilon(\cos \theta)$, according to [9], we shall have

$$(1.4) \quad \|\epsilon^{2-2s} Q_L(f, f) - Q_\epsilon(f, f)\|_{L^1} \lesssim \epsilon^{3-2s} \|f\|_{H_{\gamma+12}^5}^2,$$

where the Landau collision operator Q_L is defined as

$$Q_L(g, h)(v) \stackrel{\text{def}}{=} \nabla_v \cdot \left\{ \int_{\mathbb{R}^3} a(v - v_*) [g(v_*) \nabla_v h(v) - \nabla_v g(v_*) h(v)] dv_* \right\}.$$

Here the symmetrical matrix a is given by

$$(1.5) \quad a(v) = \Lambda |v|^{\gamma+2} \left(I - \frac{v \otimes v}{|v|^2} \right),$$

where Λ is a constant.

This motivates us to compensate the omission of grazing collisions by Landau operator. Specifically, we consider the operator

$$(1.6) \quad M^\epsilon(g, h) \stackrel{\text{def}}{=} Q^\epsilon(g, h) + \epsilon^{2-2s} Q_L(g, h),$$

and propose our approximate equation,

$$(1.7) \quad \partial_t f = M^\epsilon(f, f).$$

If \tilde{f}^ϵ is the solution to equation (1.7), we will prove

$$(1.8) \quad f = \tilde{f}^\epsilon + O(\epsilon^{3-2s}).$$

That is, by adding Landau operator to the cutoff Boltzmann equation, we increase the order of error from $2 - 2s$ to $3 - 2s$. The accuracy of approximation of the Boltzmann equation (1.1) by equation (1.7) remains even if the singularity parameter s goes to 1. Another motivation for studying equation (1.7) is the recent development of numerical methods. We believe that our approximate equation can be solved numerically. In this regard, see next subsection for a detailed discussion. We emphasize that the solutions of our approximate equation (1.7) also have the above mentioned properties, namely, conservation of mass, moment, energy and entropy dissipation.

In the current paper, we study the well-posedness of equation (1.7) and then give the error analysis of the approximate equation (1.7) and the original Boltzmann equation (1.1). There are two main difficulties in the current paper. One is to show the existence of a non-negative solution to equation (1.7). We proceed by constructing a sequence of convergent non-negative functions with its limit being the solution. Since we consider hard potentials ($\gamma > 0$), there will be an increase of weight at each iteration. Observing the coefficient before the weight increased term is strictly less 1, we prove that, on a whole level, the increased weight is limited. The other difficulty is related to the estimate of the error function F_R^ϵ as defined in (4.1). Again, weight increase problem happens here and another problem is no sign information of F_R^ϵ . We circumvent the problem of lacking sign information by writing the equation of error function in a suitable way. The weight increase problem is dealt with by carefully separating the integration region such that either the increased weight is eliminated or the coefficient before the weight increased term is controlled as desired.

1.4. Existing numerical results and future work. Our approximate equation contains both the angular cut-off Boltzmann operator Q^ϵ and Landau operator Q_L . Numerical methods of the Boltzmann equation and Landau equation have been investigated extensively. The most famous one is Kac's program. Kac started from the Markov process corresponding to collisions only, and try to prove the limit towards the spatially homogeneous Boltzmann equation. For Kac's program approximating Boltzmann equation, we refer to the recent work [15] and the references therein. In [15], the authors proved the propagation of chaos quantitatively in an abstract framework by proving stability and convergence estimates between linear semigroups. They then applied their results to prove the propagation of chaos of Kac's program in the cases of hard sphere model ($B(v - v_*, \cos \theta) = |v - v_*|$) and true Maxwell molecules ($B(v - v_*, \cos \theta) = b(\cos \theta)$).

As for particle system approximating the Landau equation, we refer to [8] and the references therein. The authors in [8] proved quantitatively the propagation of chaos for a N -particle continuous drift diffusion process under the cases of Maxwell molecules ($\gamma = 0$) and hard potentials ($0 < \gamma \leq 1$).

As one can see from above, the Boltzmann equation corresponds to the limit of jump processes, while the Landau equation corresponds to the limit of continuous processes. If we are to numerically solve our approximate equation (1.7), we need some jump-diffusion processes. Actually, the method in [15] is general and robust to deal with mixture of jump and diffusion processes. As shown to be successful in [16], the authors considered the Boltzmann equation for diffusively excited granular media, used jump-diffusion processes to approximate it, and then proved the propagation of chaos. The jump part is the Boltzmann operator with an integrable kernel, while the diffusive part is a Laplace operator. We know that the Landau operator behaves like the Laplace operator, except with some compensation to conserve energy.

In the recent work [7], the authors replaced the small collisions by a small diffusion term to approximate the Kac equation without cutoff, and successfully built a stochastic particle system to approximate the solution of the Kac equation without cutoff. The Kac equation is a one-dimensional case of the Boltzmann equation.

Thanks to the above breakthroughs, our approximate equation (1.7) has great potential to be solved numerically. In our future work, we will build a particle system based on equation (1.7) and prove the propagation of chaos.

1.5. Notations and main results. Let us introduce the function spaces and notations which we shall use throughout the paper.

- For integer $N \geq 0$, we define the Sobolev space

$$H^N = \left\{ f(v) : \sum_{|\alpha| \leq N} \|\partial_v^\alpha f\|_{L^2} < +\infty \right\},$$

where the multi-index $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3$ and $\partial_v^\alpha = \partial_{v_1}^{\alpha_1} \partial_{v_2}^{\alpha_2} \partial_{v_3}^{\alpha_3}$.

- For real number m, l , we define the weighted Sobolev space

$$H_l^m = \left\{ f(v) : \|\langle D \rangle^m \langle \cdot \rangle^l f\|_{L^2} < +\infty \right\},$$

where $\langle v \rangle \stackrel{\text{def}}{=} (1 + |v|^2)^{\frac{1}{2}}$, and $a(D)$ is the pseudo-differential operator with symbol $a(\xi)$ defined by

$$(a(D)f)(v) \stackrel{\text{def}}{=} \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} e^{i(v-u) \cdot \xi} a(\xi) f(u) du d\xi.$$

- We also introduce the standard notations

$$\|f\|_{L_q^p} = \left(\int_{\mathbb{R}^3} |f(v)|^p \langle v \rangle^{qp} dv \right)^{\frac{1}{p}}, \quad \|f\|_{L \log L} = \int_{\mathbb{R}^3} |f| \log(1 + |f|) dv.$$

- For the ease of notation, let us define a new norm $\|\cdot\|_{\epsilon, m, l}$ for any $\epsilon, l > 0$ and $m \in \mathbb{N}$ as:

$$\|f\|_{\epsilon, m, l}^2 \stackrel{\text{def}}{=} \|f\|_{H_l^{m+s}}^2 + \epsilon^{2-2s} \|f\|_{H_l^{m+1}}^2,$$

If $m = 0$, we simply write $\|\cdot\|_{\epsilon, l}$ instead of $\|\cdot\|_{\epsilon, 0, l}$. If $m = l = 0$, we simply write $\|\cdot\|_\epsilon$ instead of $\|\cdot\|_{\epsilon, 0}$. Then for any $\epsilon \leq 1$, $\|\cdot\|_{H_l^{m+s}} \leq \|\cdot\|_{\epsilon, m, l} \leq 2\|\cdot\|_{H_l^{m+1}}$.

- Let us define the symbol $W^\epsilon(\xi)$ by

$$W^\epsilon(\xi) = \langle \xi \rangle^s \mathbf{1}_{|\xi| \leq \frac{1}{\epsilon}} + \epsilon^{-s} \mathbf{1}_{|\xi| > \frac{1}{\epsilon}},$$

which comes from the coercivity estimate of the cut-off Boltzmann operator Q^ϵ .

- For any $f, g \in L^2(\mathbb{R}^3)$, we denote by $\langle f, g \rangle$ the inner product of f and g .
- By $a \lesssim b$, we mean that there is a uniform constant C , which may be different on different lines, such that $a \leq Cb$. We write $a \sim b$ if both $a \lesssim b$ and $b \lesssim a$.

We do not bother to distinguish a function and its value at a point. For example, we do not distinguish weight function $\langle \cdot \rangle^l$ and the value $\langle v \rangle^l$ it takes at a point v .

We recall Young's inequality for use in future. For $a, b \geq 0$ and $p, q > 1$, with $\frac{1}{p} + \frac{1}{q} = 1$, there holds

$$(1.9) \quad ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

As a result, for any $\eta > 0$, we have the basic inequality

$$(1.10) \quad ab \leq \eta a^p + (p\eta)^{-\frac{q}{p}} \frac{b^q}{q}.$$

We also recall the Gronwall's inequality. For any $a, b \in \mathbb{R}$, and a function y defined on \mathbb{R}_+ satisfying

$$\frac{dy}{dt} \leq a + by(t),$$

then

$$(1.11) \quad y(t) \leq y(0)e^{bt} + \frac{a}{b}(e^{bt} - 1).$$

There is also an integral type of Gronwall's inequality. Let y, α, β be functions defined on \mathbb{R}_+ . If β is nonnegative and for any $t > a \geq 0$, y satisfies

$$y(t) \leq \alpha(t) + \int_a^t \beta(r)y(r)dr,$$

then

$$(1.12) \quad y(t) \leq \alpha(t) + \int_a^t \alpha(r)\beta(r) \exp\left(\int_r^t \beta(u)du\right)dr.$$

If, in addition, the function α is non-decreasing, then

$$(1.13) \quad y(t) \leq \alpha(t) \exp\left(\int_a^t \beta(r)dr\right).$$

Before stating our main results, let us give the definition of ϕ which is related to the weight function:

$$(1.14) \quad \begin{cases} \phi(0, l) = 2l + 5; \\ \phi(s, l) = \frac{(2l + 4)(2 + s) - 2l}{s}; \\ \phi(1, l) = \max\{\phi(s, x(l)), y(l)\}; \\ \phi(m, l) = \max\{u(m, l), \phi(m - 1, z(l)), m \geq 2, \end{cases}$$

where

$$(1.15) \quad \begin{cases} x(l) = \frac{2l + 7}{s} - \frac{1 - s}{s}(l + \frac{\gamma}{2}); \\ y(l) = \frac{3x(l) - (s + 2)l}{1 - s}; \\ z(l) = 2l + 7 + \frac{l + 7}{s}; \\ u(m, l) = (m + 2)z(l) - (m + 1)l. \end{cases}$$

We begin with the first result concerns the propagation of the moments and smoothness for the solution to our approximate equation.

Theorem 1.1. *Let $\phi : \mathbb{N} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ be the function defined as in (1.14). Let $N \in \mathbb{N}$ and $l \geq 0$. If $f_0 \in L_q^1 \cap H_l^N$ with $q \geq \phi(N, l)$, then (1.7) admits a non-negative and unique solution f^ϵ in $L^\infty([0, \infty]; L_q^1 \cap H_l^N)$ and moreover there exists a constant C , depending only on $\|f_0\|_{L_q^1}$ and $\|f_0\|_{H_l^N}$, such that for any $t \geq 0$ and ϵ small enough,*

$$(1.16) \quad \|f^\epsilon(t)\|_{L_q^1} \leq C + \|f_0\|_{L_q^1};$$

$$(1.17) \quad \|f^\epsilon(t)\|_{H_l^N}^2 + \int_t^{t+1} \|f^\epsilon(r)\|_{\epsilon, N, l+\gamma/2}^2 dr \leq C(\|f_0\|_{L_q^1}, \|f_0\|_{H_l^N}).$$

Remark 1.1. *The result of Theorem 1.1 is also true when $\epsilon = 0$, which corresponds to the propagation of moments and smoothness of solution of the original Boltzmann equation (1.1).*

The last two theorems describe the error between solutions of the Boltzmann equation and our approximate equation.

Theorem 1.2. *Let $l \geq 0$ such that $(\frac{4}{\pi})^{2l-2s}(l-s) \geq \frac{2^{4-2s}\pi K}{A_2}$ and $2l \geq \frac{s}{1-s}(\gamma+2) + \gamma$. Suppose $f_0 \in L_q^1 \cap H_{2l+\gamma+12}^5$ with $q \geq \phi(5, 2l + \gamma + 12)$. Let f and f^ϵ be solutions to the Boltzmann equation (1.1) and the approximated equation (1.7) with the same initial datum f_0 respectively, then we have for any $t \geq 0$,*

$$(1.18) \quad \|f(t) - f^\epsilon(t)\|_{L_{2l}^1} \leq C(f_0, t)\epsilon^{3-2s},$$

where $C(f_0, t)$ is a constant depending only on $\|f_0\|_{L_q^1}, \|f_0\|_{H_{2l+\gamma+12}^5}$ and time t .

Let us introduce the definition of ψ :

$$(1.19) \quad \begin{cases} \psi(0, l) = 2l + \gamma + 17; \\ \psi(m, l) = l + \gamma + 10, m \geq 1. \end{cases}$$

and φ :

$$(1.20) \quad \begin{cases} \varphi(0, l) = \phi(5, 2l + \gamma + 17); \\ \varphi(m, l) = \max\{\varphi(m-1, z(l)), \rho(m, l)\}, m \geq 1. \end{cases}$$

Then we have:

Theorem 1.3. *Let $N \in \mathbb{N}$ and $l \geq 0$ such that $(\frac{4}{\pi})^{2l+5-2s}(2l+5-2s) \geq \frac{2^{5-2s}\pi K}{A_2}$ and $2l+5 \geq \frac{s}{1-s}(\gamma+2) + \gamma$. Let $\psi, \varphi : \mathbb{N} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ be functions defined as in (1.19) and (1.20). Suppose $f_0 \in L_q^1 \cap H_{\psi(N, l)}^{N+5}$ with $q \geq \varphi(N, l)$. Let f and f^ϵ be solutions to the Boltzmann equation (1.1) and the approximated equation (1.7) with the same initial datum f_0 respectively, then we have for any $t \geq 0$,*

$$(1.21) \quad \|f(t) - f^\epsilon(t)\|_{H_l^N} \leq C(f_0, t)\epsilon^{3-2s},$$

where $C(f_0, t)$ is a constant depending only on $\|f_0\|_{L^1_q}, \|f_0\|_{H^{N+5}_{\psi(N,t)}}$ and time t .

1.6. Plan of the paper. In section 2, we state three estimates (upper bound, coercivity, commutator) of the operator M^ϵ . Section 3 is devoted to the well-posedness theory of our approximate equation, namely, uniqueness and existence of non-negative solution. In the last section, we prove the high order convergence of solutions between the Boltzmann equation and our approximate equation.

2. ESTIMATES OF THE COLLISION OPERATORS

In this section, we state three estimates of the operator M^ϵ , as defined in (1.6) which will be used frequently in next sections. We begin with upper bound of the collision operator.

Theorem 2.1. *Suppose the collision kernel B satisfies the Assumption (A-1)-(A-4), and Q^ϵ is the collision operator associated to the collision kernel B^ϵ . Let $w_1, w_2 \in \mathbb{R}$ with $w_1 + w_2 \geq \gamma + 2$, $a_1, a_2 \geq 0$ with $a_1 + a_2 = 2s$ and $b_1, b_2 \geq 0$ with $b_1 + b_2 = 2$. Then for smooth functions g, h and f , the following estimate holds uniformly with respect to ϵ :*

$$(2.1) \quad |\langle M^\epsilon(g, h), f \rangle| \lesssim \|g\|_{L^1_{\gamma+2+(-w_1)^++(-w_2)^+}} (\|h\|_{H^{a_1}_{w_1}} \|f\|_{H^{a_2}_{w_2}} + \epsilon^{2-2s} \|h\|_{H^{b_1}_{w_1}} \|f\|_{H^{b_2}_{w_2}}).$$

Proof. For the cut-off Boltzmann operator Q^ϵ , as in [11], for any $w_1, w_2 \in \mathbb{R}$ with $w_1 + w_2 \geq \gamma + 2$, there holds

$$(2.2) \quad |\langle Q^\epsilon(g, h), f \rangle| \lesssim \|g\|_{L^1_{\gamma+2s+(-w_1)^++(-w_2)^+}} \|h\|_{H^{a_1}_{w_1}} \|f\|_{H^{a_2}_{w_2}}.$$

Again from [11], we have

$$(2.3) \quad |\langle Q_L(g, h), f \rangle| \lesssim \|g\|_{L^1_{\gamma+2+(-w_1)^++(-w_2)^+}} \|h\|_{H^{b_1}_{w_1}} \|f\|_{H^{b_2}_{w_2}}.$$

Patching together the above two estimates, the estimate (2.1) follows accordingly. \square

We now turn to coercivity estimate of the operator.

Theorem 2.2. *Suppose the collision kernel B satisfies the Assumption (A-1)-(A-4), and Q^ϵ is the collision operator associated to the collision kernel B^ϵ . Suppose function g is nonnegative and satisfies*

$$(2.4) \quad \|g\|_{L^1_2} + \|g\|_{L \log L} < \infty,$$

then there exists constants $C_1(g)$ and $C_2(g)$ depending only on $\|g\|_{L^1_1}$ and $\|g\|_{L \log L}$ such that

$$(2.5) \quad -\langle M^\epsilon(g, f), f \rangle \geq C_1(g) \|f\|_{L^2_{\epsilon, \gamma/2}}^2 - C_2(g) \|f\|_{L^2_{\gamma/2}}^2.$$

Proof. For the cut-off Boltzmann operator Q^ϵ , with a similar argument as in [1], one has

$$-\langle Q^\epsilon(g, f), f \rangle \geq C_1(g) \|W^\epsilon(D)f\|_{L^2_{\gamma/2}}^2 - C_2(g) \|f\|_{L^2_{\gamma/2}}^2.$$

For the Landau operator Q_L , by [6], there holds

$$(2.6) \quad -\langle Q_L(g, f), f \rangle_v \geq C_1(g) \|f\|_{H^1_{\gamma/2}}^2 - C_2(g) \|f\|_{L^2_{\gamma/2}}^2.$$

The coercivity estimate (2.5) follows by noting that

$$\|W^\epsilon(D)f\|_{L^2_{\gamma/2}}^2 + \epsilon^{2-2s} \|f\|_{H^1_{\gamma/2}}^2 \sim \|f\|_{L^2_{\epsilon, \gamma/2}}^2.$$

\square

In the last, we move to commutator estimates. We first give the commutator estimate of the cut-off Boltzmann operator Q^ϵ as a lemma.

Lemma 2.1. *Suppose the collision kernel B satisfies the Assumption (A-1)-(A-4), and Q^ϵ is the collision operator associated to the collision kernel B^ϵ . Let $N_2, N_3 \in \mathbb{R}$ and $l \geq 0$ with $N_2 + N_3 \geq l + \gamma$, and let $N_1 = |N_2| + |N_3| + \max\{|l-1|, |l-2|\}$. Then for smooth functions g, h and f , the following estimate holds uniformly with respect to ϵ :*

$$(2.7) \quad |\langle Q^\epsilon(g, h\langle v \rangle^l) - Q^\epsilon(g, h)\langle v \rangle^l, f \rangle| \lesssim \|g\|_{L^1_{N_1}} \|h\|_{H^s_{N_2}} \|f\|_{L^2_{N_3}}.$$

Proof. One may refer to [4] for a proof. \square

The next lemma is the commutator estimate of the Landau operator Q_L .

Lemma 2.2. *Let $N_2, N_3 \in \mathbb{R}$ and $l \geq 0$ with $N_2 + N_3 \geq l + \gamma$. Then for smooth functions g, h and f , the following estimate holds true:*

$$(2.8) \quad |\langle Q_L(g, h \langle v \rangle^l) - Q_L(g, h) \langle v \rangle^l, f \rangle| \leq \Lambda C(l) \|g\|_{L^1_{\gamma+3}} \|h\|_{H^1_{N_2}} \|f\|_{L^2_{N_3}},$$

where $C(l) = \max\{2l^2 + 12l, 20l - 2l^2\}$.

Proof. We define as usual the following quantities in 3-dimension:

$$b_i(z) = \sum_{j=1}^3 \partial_j a_{ij}(z) = -2\Lambda |z|^\gamma z_i, \quad c(z) = \sum_{i,j=1}^3 \partial_{ij} a_{ij}(z) = -2\Lambda(\gamma + 3) |z|^\gamma.$$

Hence the Landau operator Q_L can be rewritten as:

$$Q_L(g, h) = \sum_{i,j=1}^3 (a_{ij} * g) \partial_{ij} h - (c * g) h = \sum_{i=1}^3 \partial_i \left[\sum_{j=1}^3 (a_{ij} * g) \partial_j h - (b_i * g) h \right].$$

Then we have

$$D(g, h, f; l) \stackrel{\text{def}}{=} \langle Q_L(g, h \langle v \rangle^l) - Q_L(g, h) \langle v \rangle^l, f \rangle = \sum_{i,j=1}^3 \langle a_{ij} * g, f \partial_{ij} (h \langle v \rangle^l) - f \langle v \rangle^l \partial_{ij} h \rangle.$$

It is easy to check

$$\partial_{ij} (h \langle v \rangle^l) - \langle v \rangle^l \partial_{ij} h = l \langle v \rangle^{l-2} (v_i \partial_j h + v_j \partial_i h) + l \langle v \rangle^{l-2} [(l-2) \frac{v_i v_j}{\langle v \rangle^2} + \delta_{ij}] h.$$

Thus we have

$$\begin{aligned} D(g, h, f; l) &= l \int_{\mathbb{R}^6} g_* f \langle v \rangle^{l-2} \left[\sum_{i,j} a_{ij} (v - v_*) (v_i \partial_j h + v_j \partial_i h) \right] dv dv_* \\ &\quad + l(l-2) \int_{\mathbb{R}^6} g_* h f \langle v \rangle^{l-2} \frac{\sum_{i,j} a_{ij} (v - v_*) v_i v_j}{\langle v \rangle^2} dv dv_* \\ &\quad + l \int_{\mathbb{R}^6} g_* h f \langle v \rangle^{l-2} \sum_i a_{ii} (v - v_*) dv dv_*. \end{aligned}$$

Considering the following facts

$$\sum_{i,j=1}^3 a_{ij} (v - v_*) v_i \partial_j h = \sum_{i,j=1}^3 a_{ij} (v - v_*) v_j \partial_i h = (\nabla h)^T a (v - v_*) v = (\nabla h)^T a (v - v_*) v_*,$$

and

$$\sum_{i,j=1}^3 a_{ij} v_i v_j = \Lambda |v - v_*|^\gamma (|v|^2 |v_*|^2 - (v \cdot v_*)^2),$$

and

$$\sum_i a_{ii} = 2\Lambda |v - v_*|^{\gamma+2},$$

we arrive at

$$\begin{aligned} D(g, h, f; l) &= 2l \int_{\mathbb{R}^6} g_* f \langle v \rangle^{l-2} (\nabla h)^T a (v - v_*) v_* dv dv_* \\ &\quad + \Lambda l(l-2) \int_{\mathbb{R}^6} g_* h f \langle v \rangle^{l-2} \frac{|v - v_*|^\gamma (|v|^2 |v_*|^2 - (v \cdot v_*)^2)}{\langle v \rangle^2} dv dv_* \\ &\quad + 2\Lambda l \int_{\mathbb{R}^6} g_* h f \langle v \rangle^{l-2} |v - v_*|^{\gamma+2} dv dv_* \\ &\stackrel{\text{def}}{=} \mathfrak{I}_1 + \mathfrak{I}_2 + \mathfrak{I}_3. \end{aligned}$$

Thanks to

$$|a(v - v_*) v_*| \leq 4\Lambda \langle v_* \rangle^{\gamma+3} \langle v \rangle^{\gamma+2},$$

we have

$$|\mathfrak{I}_1| \leq 8\Lambda l \|g\|_{L_{\gamma+3}^1} \|h\|_{H_{N_2}^1} \|f\|_{L_{N_3}^2},$$

provided $N_2 + N_3 \geq l + \gamma$. Similarly, if $N_2 + N_3 \geq l + \gamma$, there holds

$$|\mathfrak{I}_3| \leq 8\Lambda l \|g\|_{L_{\gamma+2}^1} \|h\|_{L_{N_2}^2} \|f\|_{L_{N_3}^2}.$$

With the help of the fact

$$\frac{|v - v_*|^\gamma (|v|^2 |v_*|^2 - (v \cdot v_*)^2)}{\langle v \rangle^2} \leq 2 \langle v_* \rangle^{\gamma+2} \langle v \rangle^\gamma,$$

we have

$$|\mathfrak{I}_2| \leq 2\Lambda l |l - 2| \|g\|_{L_{\gamma+2}^1} \|h\|_{L_{N_2}^2} \|f\|_{L_{N_3}^2},$$

provided $N_2 + N_3 \geq l - 2 + \gamma$. Patching together the above estimates, if $N_2 + N_3 \geq l + \gamma$, we have

$$|D(g, h, f; l)| \leq \Lambda \max\{2l^2 + 12l, 20l - 2l^2\} \|g\|_{L_{\gamma+3}^1} \|h\|_{H_{N_2}^1} \|f\|_{L_{N_3}^2}.$$

□

In the end of this section, we state the commutator estimate of the operator M^ϵ .

Theorem 2.3. *Suppose the collision kernel B satisfies the Assumption (A-1)-(A-4), and Q^ϵ is the collision operator associated to the collision kernel B^ϵ . Let $N_2, N_3 \in \mathbb{R}$ and $l \geq 0$ with $N_2 + N_3 \geq l + \gamma$, and let $N_1 = \max\{|N_2| + |N_3| + \max\{|l - 1|, |l - 2|\}, \gamma + 3\}$. Then for smooth functions g, h and f , the following estimate holds uniformly with respect to ϵ :*

$$(2.9) \quad |\langle M^\epsilon(g, h \langle v \rangle^l) - M^\epsilon(g, h) \langle v \rangle^l, f \rangle| \lesssim \|g\|_{L_{N_1}^1} (\|h\|_{H_{N_2}^s} + \epsilon^{2-2s} \|h\|_{H_{N_2}^1}) \|f\|_{L_{N_3}^2}.$$

Proof. The commutator estimate (2.9) follows from lemma 2.1 and 2.2. □

3. WELL-POSEDNESS FOR APPROXIMATE EQUATION (1.7): EXISTENCE AND UNIQUENESS

In this section, we will show that (1.7) admits a non-negative, unique and smooth solution if the initial data is smooth. To do that, we separate the proof into three steps. In the first step, we prove that the linear equation to (1.7) admits a non-negative and smooth solution. Then in the next step, by using Picard iteration scheme, we get the well-posedness result. In the final step, we improve the well-posedness result by applying the symmetric property of the collision operators.

3.1. Well-posedness of linear equation to (1.7). Throughout this subsection, $\epsilon > 0$ is a fixed but small enough number. In the following, we construct a non-negative solution to the linear equation:

$$(3.1) \quad \begin{cases} \partial_t f = Q^\epsilon(g, f) + \epsilon^{2-2s} Q_L(g, f) \\ f|_{t=0} = f_0. \end{cases}$$

Let us define two operators:

$$Q^{\epsilon+}(g, h) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} \int_{SS^2} B^\epsilon(v - v_*, \sigma) g'_* h' d\sigma dv_*,$$

$$Q^{\epsilon-}(g, h) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} \int_{SS^2} B^\epsilon(v - v_*, \sigma) g_* h d\sigma dv_* = \mathcal{L}(g)h.$$

Then we have $Q^\epsilon = Q^{\epsilon+} - Q^{\epsilon-}$, so we call $Q^{\epsilon+}$ the gain operator and $Q^{\epsilon-}$ the loss operator.

We first give a proposition, which shall be used in both the current section and the next section.

Proposition 3.1. *Let $p \geq 2, n = \frac{v-v_*}{|v-v_*|}, u = \frac{v+v_*}{|v+v_*|}, j = \frac{u-(u \cdot n)n}{|u-(u \cdot n)n|}, h = \sqrt{|v|^2 |v_*|^2 - (v \cdot v_*)^2}$, and $E(\theta) = \langle v \rangle^2 \cos^2 \frac{\theta}{2} + \langle v_* \rangle^2 \sin^2 \frac{\theta}{2}$. Suppose ω is the vector such that $\sigma = \cos \theta n + \sin \theta \omega$, then there holds*

$$(3.2) \quad \begin{aligned} \langle v' \rangle^{2p} - \langle v \rangle^{2p} &\leq -\langle v \rangle^{2p} (1 - \cos^{2p} \frac{\theta}{2}) + \langle v_* \rangle^{2p} \sin^{2p} \frac{\theta}{2} + p(E(\theta))^{p-1} h(j \cdot \omega) \sin \theta \\ &\quad + (\frac{1}{2} \max\{2^{p-3}, 1\} p(p-1) + 2^{p-1}) \langle v_* \rangle^{2p-2} \langle v \rangle^{2p-2} \sin^2 \theta. \end{aligned}$$

Proof. It is easy to check $\langle v' \rangle^2 = E(\theta) + h(j \cdot \omega) \sin \theta$. By Taylor expansion, we have

$$\begin{aligned} \langle v' \rangle^{2p} &= (E(\theta))^p + p(E(\theta))^{p-1} h(j \cdot \omega) \sin \theta \\ &\quad + p(p-1)(h(j \cdot \omega) \sin \theta)^2 \int_0^1 (1-\kappa)(E(\theta) + \kappa h(j \cdot \omega) \sin \theta)^{p-2} d\kappa. \\ &\stackrel{\text{def}}{=} \mathfrak{M}_1 + \mathfrak{M}_2 + \mathfrak{M}_3. \end{aligned}$$

For the last term \mathfrak{M}_3 , we have for any $\kappa \in [0, 1]$:

$$\begin{aligned} E(\theta) + \kappa h(j \cdot \omega) \sin \theta &\leq (\langle v \rangle^2 + \langle v_* \rangle^2) \left(1 - \frac{1-\kappa}{4} \sin^2 \theta\right) \\ &\leq \langle v \rangle^2 + \langle v_* \rangle^2. \end{aligned}$$

Together with $h^2 \leq \langle v \rangle^2 \langle v_* \rangle^2$, we arrive at

$$\begin{aligned} \mathfrak{M}_3 &\leq p(p-1) \langle v \rangle^2 \langle v_* \rangle^2 (\langle v \rangle^2 + \langle v_* \rangle^2)^{p-2} \sin^2 \theta \int_0^1 (1-\kappa) d\kappa \\ &\leq \frac{1}{2} \max\{2^{p-3}, 1\} p(p-1) \langle v \rangle^{2p-2} \langle v_* \rangle^{2p-2} \sin^2 \theta. \end{aligned}$$

For the term \mathfrak{M}_1 , we have

$$\begin{aligned} (3.3) \quad &(\langle v \rangle^2 \cos^2 \frac{\theta}{2} + \langle v_* \rangle^2 \sin^2 \frac{\theta}{2})^p \\ &\leq \sum_{k=1}^{k_p} \binom{p}{k} \left\{ \langle v \rangle^{2k} \cos^{2k} \frac{\theta}{2} \langle v_* \rangle^{2(p-k)} \sin^{2(p-k)} \frac{\theta}{2} + \langle v \rangle^{2(p-k)} \cos^{2(p-k)} \frac{\theta}{2} \langle v_* \rangle^{2k} \sin^{2k} \frac{\theta}{2} \right\} \\ &\leq \langle v \rangle^{2p} \cos^{2p} \frac{\theta}{2} + \langle v_* \rangle^{2p} \sin^{2p} \frac{\theta}{2} + 2^p \langle v \rangle^{2p-2} \langle v_* \rangle^{2p-2} \sin^2 \frac{\theta}{2}. \end{aligned}$$

Combining $\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M}_3$, we arrive at (3.2). \square

We begin with an equation which shall be used to construct solution to the linear equation to (3.1).

Lemma 3.1. *Let $g, h \geq 0$ be smooth functions. Suppose f^ϵ is the solution to the following equation*

$$(3.4) \quad \begin{cases} \partial_t f = Q^{\epsilon+}(g, h) - Q^{\epsilon-}(g, f) + \epsilon^{2-2s} Q_L(g, f) \\ f|_{t=0} = f_0 \geq 0. \end{cases}$$

Then $f^\epsilon(t) \geq 0$ for any $t \geq 0$.

Proof. Denote $f_-^\epsilon = \min\{0, f^\epsilon\} \leq 0$, then we have $f_-^\epsilon|_{t=0} = 0$, and

$$\frac{d}{dt} \left(\frac{1}{2} \|f_-^\epsilon\|_{L^2}^2 \right) + \int_{\mathbb{R}^3} \mathcal{L}(g)(f_-^\epsilon)^2 dv = \int_{\mathbb{R}^3} Q^{\epsilon+}(g, h) f_-^\epsilon dv + \epsilon^{2-2s} \langle Q_L(g, f^\epsilon), f_-^\epsilon \rangle.$$

Since $g, h \geq 0$ and $f_-^\epsilon \leq 0$, it is clear that

$$\int_{\mathbb{R}^3} Q^{\epsilon+}(g, h) f_-^\epsilon dv \leq 0.$$

By the definition of Q_L , we have

$$\begin{aligned} \langle Q_L(g, f^\epsilon), f_-^\epsilon \rangle &= - \int_{\mathbb{R}^6} g_*(\nabla f_-^\epsilon)^T a(v - v_*) \nabla f_-^\epsilon dv dv_* \\ &\quad + \Lambda(\gamma + 3) \int_{\mathbb{R}^6} |v - v_*|^\gamma g_*(f_-^\epsilon)^2 dv dv_* \\ &\stackrel{\text{def}}{=} \mathfrak{J}_1 + \mathfrak{J}_2. \end{aligned}$$

Since a is a positive semi-definite matrix, we have $\mathfrak{J}_1 \leq 0$. By assumption (A-2), there holds $\int_{S^{s^2}} b^\epsilon(\cos \theta) d\sigma \sim \frac{\epsilon^{-2s}}{s}$. Therefore, there exists $\epsilon_* > 0$ such that, for any $0 < \epsilon \leq \epsilon_*$,

$$\epsilon^{2-2s} \mathfrak{J}_2 \leq \frac{1}{2} \int_{\mathbb{R}^3} \mathcal{L}(g)(f_-^\epsilon)^2 dv.$$

Finally, we arrive at

$$\frac{d}{dt} \left(\frac{1}{2} \|f_-^\epsilon\|_{L^2}^2 \right) + \frac{1}{2} \int_{\mathbb{R}^3} \mathcal{L}(g)(f_-^\epsilon)^2 dv \leq 0.$$

Thus $\|f_-^\epsilon(t)\|_{L^2} = 0$ for any $t \geq 0$, which implies $f^\epsilon(t) \geq 0$ for any $t \geq 0$. \square

Now we are ready to construct a solution to the linear equation (1.7).

Lemma 3.2. *Let $l \geq 4, T > 0$ be real numbers. Suppose the non-negative datum $f_0 \in H_{l+3\gamma/2+10}^5 \cap L_{l+5\gamma/2+16}^1$ with $\|f_0\|_{L^1} > 0$. Suppose $g(t, v)$ is a non-negative function satisfying*

$$M = \sup_{0 \leq t \leq T} \|g(t)\|_{H_{2l+3\gamma+22}^5 \cap L_{l+5\gamma/2+16}^1} + \int_0^T \|g(t)\|_{L_{l+7\gamma/2+16}^1} dt < \infty \text{ and } m = \inf_{0 \leq t \leq T} \|g(t)\|_{L^1} > 0,$$

then (3.1) admits a unique non-negative solution f in $L^\infty([0, T]; L_{l+5\gamma/2+16}^1 \cap H_{l+\gamma+10}^5) \cap L^1([0, T]; L_{l+7\gamma/2+16}^1)$.

Proof. Define a sequence of approximate solutions $\{f^n\}_{n \in \mathbb{N}}$ by

$$(3.5) \quad \begin{cases} f^0(t) = f_0, \text{ for any } t \geq 0; \\ \partial_t f^n = Q^{\epsilon+}(g, f^{n-1}) - Q^{\epsilon-}(g, f^n) + \epsilon^{2-2s} Q_L(g, f^n), \quad n \geq 1 \\ f^n|_{t=0} = f_0. \end{cases}$$

According to the previous lemma, we have $f^n \geq 0$.

Step 1: (Uniform Upper Bound)

Step 1.1: (Uniform Upper Bound in L_l^1)

In this step, we shall use the energy method to get the uniform upper bound of L_l^1 norm of $\{f^n\}_n$ with respect to n . Applying the basic inequality (1.10), for any $\eta > 0$, there holds

$$\begin{aligned} |v - v_*|^\gamma &\leq (|v|^2 + 2|v||v_*| + |v_*|^2)^{\frac{\gamma}{2}} \leq ((1 + \eta)|v|^2 + (1 + \frac{1}{\eta})|v_*|^2)^{\frac{\gamma}{2}} \\ &\leq (1 + \eta)^{\frac{\gamma}{2}} \langle v \rangle^\gamma + (1 + \frac{1}{\eta})^{\frac{\gamma}{2}} \langle v_* \rangle^\gamma. \end{aligned}$$

Also one has

$$(3.6) \quad \langle v' \rangle^l \leq (1 + |v|^2 + |v_*|^2)^{\frac{l}{2}} \leq \langle v_* \rangle^l + 2^l \langle v \rangle^{l-2} \langle v_* \rangle^{l-2} + \langle v \rangle^l.$$

Thanks to the above two facts, we obtain

$$\begin{aligned} (3.7) \quad \int_{\mathbb{R}^3} Q^{\epsilon+}(g, f^{n-1})(v) \langle v \rangle^l dv &= \int_{\mathbb{R}^6} \int_{S^2} b^\epsilon(\cos \theta) |v - v_*|^\gamma g_* f^{n-1} \langle v' \rangle^l dv dv_* d\sigma \\ &\leq (1 + \eta)^{\frac{\gamma}{2}} A^\epsilon \|g\|_{L^1} \|f^{n-1}\|_{L_{l+\gamma}^1} \\ &\quad + (1 + \frac{1}{\eta})^{\frac{\gamma}{2}} A^\epsilon \|g\|_{L_{l+\gamma}^1} \|f^{n-1}\|_{L^1} \\ &\quad + C(l, \gamma, \eta) A^\epsilon \|g\|_{L_l^1} \|f^{n-1}\|_{L_l^1}, \end{aligned}$$

where $A^\epsilon = \int_{S^2} b^\epsilon(\cos \theta) d\sigma$. It is easy to check

$$\begin{aligned} \langle v \rangle^\gamma &= (1 + |v - v_* + v_*|^2)^{\frac{\gamma}{2}} \leq (1 + (1 + \frac{1}{\eta})|v_*|^2 + (1 + \eta)|v - v_*|^2)^{\frac{\gamma}{2}} \\ &\leq (1 + \frac{1}{\eta})^{\gamma/2} \langle v_* \rangle^\gamma + (1 + \eta)^{\gamma/2} |v - v_*|^\gamma. \end{aligned}$$

That is, for any $\eta > 0$, there holds

$$(3.8) \quad |v - v_*|^\gamma \geq \frac{\langle v \rangle^\gamma}{(1 + \eta)^{\gamma/2}} - \eta^{-\gamma/2} \langle v_* \rangle^\gamma.$$

Then we obtain

$$(3.9) \quad \int_{\mathbb{R}^3} Q^{\epsilon-}(g, f^n)(v) \langle v \rangle^l dv \geq \frac{A^\epsilon}{(1 + \eta)^{\gamma/2}} \|g\|_{L^1} \|f^n\|_{L_{l+\gamma}^1} - \eta^{-\gamma/2} A^\epsilon \|g\|_{L_\gamma^1} \|f^n\|_{L_l^1}.$$

For the Landau operator, referring to [6], there holds

$$\begin{aligned} (3.10) \quad \int_{\mathbb{R}^3} Q_L(g, f^n)(v) \langle v \rangle^l dv &\leq l\Lambda \int_{\mathbb{R}^3} g_* f^n |v - v_*|^\gamma \langle v \rangle^{l-2} (-2|v|^2 + l|v_*|^2) dv dv_* \\ &\leq -l\Lambda \|g\|_{L^1} \|f^n\|_{L_{l+\gamma}^1} + (4l + 2)l\Lambda \|g\|_{L_4^1} \|f^n\|_{L_l^1}. \end{aligned}$$

Patching together the above estimates, we arrive at

$$\begin{aligned} \frac{d}{dt} \|f^n\|_{L_l^1} &\leq -\left(\frac{A^\epsilon}{(1+\eta)^{\gamma/2}} + \epsilon^{2-2s} l \Lambda\right) \|g\|_{L^1} \|f^n\|_{L_{l+\gamma}^1} + (1+\eta)^{\frac{\gamma}{2}} A^\epsilon \|g\|_{L^1} \|f^{n-1}\|_{L_{l+\gamma}^1} \\ &\quad + (1+\frac{1}{\eta})^{\frac{\gamma}{2}} A^\epsilon \|g\|_{L_{l+\gamma}^1} \|f^{n-1}\|_{L^1} + C(l, \gamma, \eta) A^\epsilon \|g\|_{L_l^1} \|f^{n-1}\|_{L_l^1} \\ &\quad + \eta^{-\gamma/2} A^\epsilon \|g\|_{L_\gamma^1} \|f^n\|_{L_l^1} + \epsilon^{2-2s} (4l+2) l \Lambda \|g\|_{L_4^1} \|f^n\|_{L_l^1} \end{aligned}$$

Observing that

$$\lim_{\eta \downarrow 0} \left\{ (1+\eta)^{\frac{\gamma}{2}} A^\epsilon - \frac{A^\epsilon}{(1+\eta)^{\gamma/2}} \right\} = 0,$$

thus we can take an $\eta > 0$ small enough such that,

$$(1+\eta)^{\frac{\gamma}{2}} A^\epsilon \leq \frac{A^\epsilon}{(1+\eta)^{\gamma/2}} + \frac{1}{2} \epsilon^{2-2s} l \Lambda.$$

With such a small η , let us denote $a = \frac{A^\epsilon}{(1+\eta)^{\gamma/2}} + \frac{1}{2} \epsilon^{2-2s} l \Lambda$, $\delta = \frac{1}{2} \epsilon^{2-2s} l \Lambda$, $K_1 = (1+\frac{1}{\eta})^{\frac{\gamma}{2}} A^\epsilon$, $K_2 = \sup_{0 \leq s \leq T} C(l, \gamma, \eta) A^\epsilon \|g(s)\|_{L_l^1}$, $K_3 = \sup_{0 \leq s \leq T} \{\eta^{-\gamma/2} A^\epsilon \|g(s)\|_{L_\gamma^1} + \epsilon^{2-2s} (4l+2) l \Lambda \|g(s)\|_{L_4^1}\}$. Therefore, we arrive at a neater inequality on the interval $[0, T]$,

$$\begin{aligned} \frac{d}{dt} \|f^n\|_{L_l^1} + (a+\delta) \|g\|_{L^1} \|f^n\|_{L_{l+\gamma}^1} &\leq a \|g\|_{L^1} \|f^{n-1}\|_{L_{l+\gamma}^1} + (K_1 \|g\|_{L_{l+\gamma}^1} + K_2) \|f^{n-1}\|_{L_l^1} \\ &\quad + K_3 \|f^n\|_{L_l^1}. \end{aligned}$$

By defining $y^n(t) = e^{-K_3 t} \|f^n(t)\|_{L_l^1}$ and $x^n(t) = \int_0^t e^{-K_3 s} \|g(s)\|_{L^1} \|f^n(s)\|_{L_{l+\gamma}^1} ds$ for any $0 \leq t \leq T$ and $n \geq 0$, we derive that

$$y^n(t) + (a+\delta)x^n(t) \leq \|f_0\|_{L_l^1} + a x^{n-1}(t) + \int_0^t (K_1 \|g(s)\|_{L_{l+\gamma}^1} + K_2) y^{n-1}(s) ds.$$

Now denote $S^n(t) = \sum_{i=0}^n (\frac{a}{a+\delta})^i y^{n-i}(t)$ for $n \geq 0$, by recursive derivation and noting that $y^0(t) \leq \|f_0\|_{L_l^1}$ and $x^0(t) \leq M \frac{1-e^{-K_3 t}}{K_3} \|f_0\|_{L_{l+\gamma}^1}$, we obtain

$$\begin{aligned} S^n(t) + (a+\delta)x^n(t) &\leq \sum_{i=0}^{n-1} \left(\frac{a}{a+\delta}\right)^i \|f_0\|_{L_l^1} + \left(\frac{a}{a+\delta}\right)^n y^0(t) + \left(\frac{a}{a+\delta}\right)^{n-1} a x^0(t) \\ &\quad + \int_0^t (K_1 \|g(t_{n-1})\|_{L_{l+\gamma}^1} + K_2) S^{n-1}(t_{n-1}) dt_{n-1} \\ &\leq \left(\frac{a}{\delta} + 1\right) \|f_0\|_{L_l^1} + a M \frac{1-e^{-K_3 t}}{K_3} \|f_0\|_{L_{l+\gamma}^1} \left(\frac{a}{a+\delta}\right)^{n-1} \\ &\quad + \int_0^t (K_1 \|g(t_{n-1})\|_{L_{l+\gamma}^1} + K_2) S^{n-1}(t_{n-1}) dt_{n-1}. \end{aligned}$$

By further recursive derivation, we have

$$\begin{aligned} &S^n(t) + (a+\delta)x^n(t) \\ &\leq \left(\frac{a}{\delta} + 1\right) \|f_0\|_{L_l^1} \sum_{i=0}^{n-1} \frac{(\int_0^t (K_1 \|g(s)\|_{L_{l+\gamma}^1} + K_2) ds)^i}{i!} \\ &\quad + a M \frac{1-e^{-K_3 t}}{K_3} \|f_0\|_{L_{l+\gamma}^1} \left(\frac{a}{a+\delta}\right)^{n-1} \sum_{i=0}^{n-1} \frac{(\frac{a+\delta}{a} \int_0^t (K_1 \|g(s)\|_{L_{l+\gamma}^1} + K_2) ds)^i}{i!} \\ &\quad + \int_0^t \int_0^{t_{n-1}} \cdots \int_0^{t_1} S^0(t_0) \prod_{i=0}^{n-1} (K_1 \|g(t_i)\|_{L_{l+\gamma}^1} + K_2) dt_{n-1} dt_{n-2} \cdots dt_0 \\ &\leq \left(\frac{a}{\delta} + 1\right) \|f_0\|_{L_l^1} \exp\left(\int_0^t (K_1 \|g(s)\|_{L_{l+\gamma}^1} + K_2) ds\right) \\ &\quad + a M \frac{1-e^{-K_3 t}}{K_3} \|f_0\|_{L_{l+\gamma}^1} \left(\frac{a}{a+\delta}\right)^{n-1} \exp\left(\frac{a+\delta}{a} \int_0^t (K_1 \|g(s)\|_{L_{l+\gamma}^1} + K_2) ds\right). \end{aligned}$$

Noting that

$$\|f^n(t)\|_{L_t^1} \leq e^{K_3 t} S^n(t),$$

and

$$\int_0^t \|f^n(s)\|_{L_{t+\gamma}^1} ds \leq m^{-1} e^{K_3 t} x^n(t),$$

and recalling the definition of constants K_1, K_2, K_3 , we obtain

$$\begin{aligned} (3.11) \quad & \sup_n (\|f^n(t)\|_{L_t^1} + \int_0^t \|f^n(s)\|_{L_{t+\gamma}^1} ds) \\ & \leq C(\|f_0\|_{L_{t+\gamma}^1}, t, \sup_{0 \leq s \leq t} \|g(s)\|_{L_t^1}, \int_0^t \|g(s)\|_{L_{t+\gamma}^1} ds). \end{aligned}$$

Step 1.2: (Uniform Upper Bound in L_t^2)

In this step, we show the uniform upper bound of L_t^2 norm of $\{f^n\}_n$ with respect to n . It is easy to check that

$$\begin{aligned} (3.12) \quad \frac{d}{dt} \left(\frac{1}{2} \|f^n\|_{L_t^2}^2 \right) &= \langle Q^{\epsilon+}(g, f^{n-1}) - Q^{\epsilon-}(g, f^n) + \epsilon^{2-2s} Q_L(g, f^n), f^n \langle v \rangle^{2l} \rangle \\ &\stackrel{\text{def}}{=} \mathfrak{I}_1 - \mathfrak{I}_2 + \epsilon^{2-2s} \mathfrak{I}_3. \end{aligned}$$

By Cauchy-Schwartz inequality, there holds

$$\begin{aligned} (3.13) \quad \mathfrak{I}_1 &= \int B^\epsilon g_* f^{n-1} f^n \langle v' \rangle^{2l} dv dv_* d\sigma \\ &\lesssim \left(\int B^\epsilon g_* (f^{n-1})^2 \langle v' \rangle^{2l} dv dv_* d\sigma \right)^{1/2} \times \left(\int B^\epsilon g_* (f^n)^2 \langle v' \rangle^{2l} dv dv_* d\sigma \right)^{1/2} \\ &\lesssim (A^\epsilon \|g\|_{L_{2l+\gamma}^1} \|f^{n-1}\|_{L_{l+\gamma/2}^2})^{1/2} \times (A^\epsilon \|g\|_{L^1} \|f^n\|_{L_{l+\gamma/2}^2})^{1/2} \\ &\lesssim A^\epsilon \|g\|_{L_{2l+\gamma}^1} \|f^{n-1}\|_{L_{l+\gamma/2}^2} \|f^n\|_{L_{l+\gamma/2}^2}, \end{aligned}$$

where we have used the estimate (3.6) and the usual change of variable $v \rightarrow v'$. By direct calculation, we have

$$(3.14) \quad \mathfrak{I}_2 = \int B^\epsilon g_* (f^n)^2 \langle v \rangle^{2l} dv dv_* d\sigma \leq A^\epsilon \|g\|_{L_\gamma^1} \|f^n\|_{L_{l+\gamma/2}^2}^2.$$

By coercivity estimate (2.6) and commutator (2.8) estimate of the Landau operator, thus we have

$$\begin{aligned} (3.15) \quad \mathfrak{I}_3 &= \langle Q_L(g, f^n \langle v \rangle^l), f^n \langle v \rangle^l \rangle + \{ \langle Q_L(g, f^n) \langle v \rangle^l - Q_L(g, f^n \langle v \rangle^l), f^n \langle v \rangle^l \} \\ &\leq -C_1(g) \|f^n\|_{H_{l+\gamma/2}^1}^2 + C_2(g) \|f^n\|_{L_{l+\gamma/2}^2}^2 + \Lambda C(l) \|g\|_{L_{\gamma+3}^1} \|f^n\|_{H_{l+\gamma/2}^1} \|f^n\|_{L_{l+\gamma/2}^2}. \end{aligned}$$

Now patching together the inequalities (3.13), (3.14) and (3.15), and using the basic inequality (1.10), we have

$$\frac{d}{dt} \left(\frac{1}{2} \|f^n\|_{L_t^2}^2 \right) + \frac{C_1}{2} \epsilon^{2-2s} \|f^n\|_{H_{l+\gamma/2}^1}^2 \leq \frac{C_1}{8} \epsilon^{2-2s} \|f^{n-1}\|_{L_{l+\gamma/2}^2}^2 + K_1 \|f^n\|_{L_{l+\gamma/2}^2}^2,$$

where C_1, K_1 are some positive constants depending on m, M, ϵ . For any $\lambda, s > 0$, one has

$$(3.16) \quad \|f\|_{L^2}^2 \leq \lambda \|f\|_{H^s}^2 + \frac{4\pi}{3} \lambda^{-\frac{3}{2s}} \|f\|_{L^1}^2.$$

With the help of the above inequality, we have

$$\frac{d}{dt} \|f^n\|_{L_t^2}^2 + \frac{C_1}{2} \epsilon^{2-2s} \|f^n\|_{H_{l+\gamma/2}^1}^2 \leq \frac{C_1}{4} \epsilon^{2-2s} \|f^{n-1}\|_{L_{l+\gamma/2}^2}^2 + K_1 \|f^n\|_{L_{l+\gamma/2}^2}^2,$$

for some new constant K_1 . By the previous step, with the uniform upper bound of $\|f^n\|_{L_{l+\gamma/2}^2}$, we have

$$(3.17) \quad \frac{d}{dt} \|f^n\|_{L_t^2}^2 + \frac{C_1}{2} \epsilon^{2-2s} \|f^n\|_{L_{l+\gamma/2}^2}^2 \leq \frac{C_1}{4} \epsilon^{2-2s} \|f^{n-1}\|_{L_{l+\gamma/2}^2}^2 + K_1 K_2,$$

where K_2 is some constant depending on $\|f_0\|_{L^1_{l+3\gamma/2}}$ and uniform upper bound of $\|g\|_{L^1_{l+3\gamma/2}}$. Now we use the same technique as in the previous step. Integrating both sides with respect to time, for any $t_n \in [0, t]$, we obtain

$$\|f^n(t_n)\|_{L^2_l}^2 + \frac{C_1}{2}\epsilon^{2-2s} \int_0^{t_n} \|f^n(r)\|_{L^2_{l+\gamma/2}}^2 dr \leq \|f_0\|_{L^2_l}^2 + \frac{C_1}{4}\epsilon^{2-2s} \int_0^{t_n} \|f^{n-1}(r)\|_{L^2_{l+\gamma/2}}^2 dr + K_1 K_2 t.$$

Now denote $S^n(t_n) = \sum_{i=0}^n (\frac{1}{2})^i \|f^{n-i}(t_n)\|_{L^2_l}^2$ and $x^n(t_n) = C_1 \epsilon^{2-2s} \int_0^{t_n} \|f^n(r)\|_{L^2_{l+\gamma/2}}^2 dr$ for $n \geq 0$, by recursive derivation and noting that $x^0(t_n) \leq C_1 \epsilon^{2-2s} t \|f_0\|_{L^2_{l+\gamma/2}}^2$, we obtain, for $n \geq 1$,

$$\begin{aligned} S^n(t_n) + \frac{1}{2} x^n(t_n) &\leq \sum_{i=0}^n (\frac{1}{2})^i (\|f_0\|_{L^2_l}^2 + K_1 K_2 t) + \frac{C_1}{2^{n+1}} \epsilon^{2-2s} t \|f_0\|_{L^2_{l+\gamma/2}}^2 \\ &\leq 2\|f_0\|_{L^2_l}^2 + 2K_1 K_2 t + \frac{C_1}{4} \epsilon^{2-2s} t \|f_0\|_{L^2_{l+\gamma/2}}^2. \end{aligned}$$

By tracking the definitions of constants K_1, K_2 , we obtain

$$(3.18) \quad \sup_{0 \leq s \leq t} \sup_n \|f^n(s)\|_{L^2_l} \leq C(\|f_0\|_{L^1_{l+3\gamma/2}}, \|f_0\|_{L^2_{l+\gamma/2}}, t, \sup_{0 \leq s \leq t} \|g(s)\|_{L^2_{2l+\gamma+2}}).$$

Step 1.3: (Uniform Upper Bound in H^m_l with $m \geq 1$)

Fix an α with $|\alpha| \leq m$, one has

$$\partial_t \partial_v^\alpha f^n = \sum_{\alpha_1 + \alpha_2 = \alpha} \binom{\alpha}{\alpha_1} [Q^{\epsilon+}(\partial_v^{\alpha_1} g, \partial_v^{\alpha_2} f^{n-1}) - Q^{\epsilon-}(\partial_v^{\alpha_1} g, \partial_v^{\alpha_2} f^n) + \epsilon^{2-2s} Q_L(\partial_v^{\alpha_1} g, \partial_v^{\alpha_2} f^n)].$$

Then we have

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|\partial_v^\alpha f^n\|_{L^2_l}^2 \right) &= \sum_{\alpha_1 + \alpha_2 = \alpha} \binom{\alpha}{\alpha_1} [\langle Q^{\epsilon+}(\partial_v^{\alpha_1} g, \partial_v^{\alpha_2} f^{n-1}), \partial_v^\alpha f^n \rangle_{L^2_l} \\ &\quad - \langle Q^{\epsilon-}(\partial_v^{\alpha_1} g, \partial_v^{\alpha_2} f^n), \partial_v^\alpha f^n \rangle_{L^2_l} + \epsilon^{2-2s} \langle Q_L(\partial_v^{\alpha_1} g, \partial_v^{\alpha_2} f^n), \partial_v^\alpha f^n \rangle_{L^2_l}] \\ &\stackrel{\text{def}}{=} \sum_{\alpha_1 + \alpha_2 = \alpha} \binom{\alpha}{\alpha_1} [\mathfrak{I}_1(\alpha_1, \alpha_2) - \mathfrak{I}_2(\alpha_1, \alpha_2) + \epsilon^{2-2s} \mathfrak{I}_3(\alpha_1, \alpha_2)]. \end{aligned}$$

As the same as (3.13), we have

$$|\mathfrak{I}_1(\alpha_1, \alpha_2)| \lesssim A^\epsilon \|g\|_{H^m_{2l+\gamma+2}} \|f^{n-1}\|_{H^m_{l+\gamma/2}} \|f^n\|_{H^m_{l+\gamma/2}}$$

As the same as (3.14), we have

$$|\mathfrak{I}_2(\alpha_1, \alpha_2)| \lesssim A^\epsilon \|g\|_{H^m_{\gamma+2}} \|f^n\|_{H^m_{l+\gamma/2}}^2.$$

When $|\alpha_2| \leq |\alpha| - 1 \leq m - 1$, by upper bound estimate (2.3) and commutator (2.8) estimate of the Landau operator, we have

$$\begin{aligned} |\mathfrak{I}_3(\alpha_1, \alpha_2)| &\leq |\langle Q_L(\partial_v^{\alpha_1} g, \partial_v^{\alpha_2} f^n \langle v \rangle^l), \partial_v^\alpha f^n \langle v \rangle^l \rangle| \\ &\quad + |\langle \{Q_L(\partial_v^{\alpha_1} g, f^n) \langle v \rangle^l - Q_L(\partial_v^{\alpha_1} g, \partial_v^{\alpha_2} f^n \langle v \rangle^l), \partial_v^\alpha f^n \langle v \rangle^l \} \rangle| \\ &\lesssim \|g\|_{H^m_{\gamma+4}} \|f^n\|_{H^m_{l+\gamma/2+2}} \|f^n\|_{H^{m+1}_{l+\gamma/2}} + \|g\|_{H^m_{\gamma+5}} \|f^n\|_{H^m_{l+\gamma/2}}^2. \end{aligned}$$

When $\alpha_2 = \alpha$, as the same as (3.15), we have

$$\mathfrak{I}_3(0, \alpha) \leq -C_1(g) \|\partial_v^\alpha f^n\|_{H^1_{l+\gamma/2}}^2 + C_2(g) \|\partial_v^\alpha f^n\|_{L^2_{l+\gamma/2}}^2 + \Lambda C(l) \|g\|_{H^m_{\gamma+5}} \|f^n\|_{H^{m+1}_{l+\gamma/2}} \|f^n\|_{H^m_{l+\gamma/2}}$$

Now patching together the above estimates and taking sum over $|\alpha| \leq m$, we have

$$\frac{1}{2} \frac{d}{dt} \|f^n\|_{H^m_l}^2 + \frac{C_1}{2} \epsilon^{2-2s} \|f^n\|_{H^{m+1}_{l+\gamma/2}}^2 \leq \frac{C_1}{4} \epsilon^{2-2s} \|f^{n-1}\|_{H^m_{l+\gamma/2}}^2 + K_1 \|f^n\|_{H^m_{l+\gamma/2+2}}^2,$$

where C_1, K_1 are some positive constants depending on uniform upper bound of $\|g\|_{H^m_{2l+\gamma+2}}$ and uniform lower bound of $\|g\|_{L^1}$. Thanks to interpolation theory and the basic inequality (1.10), for any $\eta > 0$, there exists some constant C_η such that

$$\|f^n\|_{H^m_{l+\gamma/2+2}}^2 \leq \eta \|f^n\|_{H^{m+1}_{l+\gamma/2}}^2 + C_\eta \|f^n\|_{L^1_{l+\gamma/2+2m+6}}^2,$$

thus we have

$$\frac{d}{dt} \|f^n\|_{H_l^m}^2 + \frac{C_1}{2} \epsilon^{2-2s} \|f^n\|_{H_{l+\gamma/2}^m}^2 \leq \frac{C_1}{8} \epsilon^{2-2s} \|f^{n-1}\|_{H_{l+\gamma/2}^m}^2 + K_1 \|f^n\|_{L_{l+\gamma/2+2m+6}^1}^2,$$

for some new constant K_1 . By the previous step, with the uniform upper bound of $\|f^n\|_{L_{l+\gamma/2+2m+6}^1}$, we have

$$(3.19) \quad \frac{d}{dt} \|f^n\|_{L_l^2}^2 + \frac{C_1}{2} \epsilon^{2-2s} \|f^n\|_{L_{l+\gamma/2}^2}^2 \leq \frac{C_1}{4} \epsilon^{2-2s} \|f^{n-1}\|_{L_{l+\gamma/2}^2}^2 + K_1 K_2,$$

where K_2 is some constant depending on $\|f_0\|_{L_{l+3\gamma/2+2m+6}^1}$ and uniform upper bound of $\|g\|_{L_{l+3\gamma/2+2m+6}^1}$. Noticing that inequality (3.19) has exactly the same structure as inequality (3.17), we have

$$(3.20) \quad \sup_{0 \leq s \leq t} \sup_n \|f^n(s)\|_{H_l^m} \leq C(\|f_0\|_{L_{l+3\gamma/2+2m+6}^1}, \|f_0\|_{H_{l+\gamma/2}^m}, t, \sup_{0 \leq s \leq t} \|g(s)\|_{H_{2l+\gamma+2}^m}, \sup_{0 \leq s \leq t} \|g(s)\|_{L_{l+3\gamma/2+2m+6}^1}).$$

Step 2: (Cauchy Sequence)

In this step, we prove that $\{f^n(t)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in L_l^1 for any $t \geq 0$. Set $h^n = f^n - f^{n-1}$ for $n \geq 1$. Then for $n \geq 2$, we have

$$(3.21) \quad \begin{cases} \partial_t h^n = Q^{\epsilon+}(g, h^{n-1}) - Q^{\epsilon-}(g, h^n) + \epsilon^{2-2s} Q_L(g, h^n), \\ h^n|_{t=0} = 0. \end{cases}$$

Because we are uncertain about the sign of h^n , we have to introduce the sign function $\text{sgn}(h^n)$. Similar as in (3.7), we obtain

$$(3.22) \quad \begin{aligned} & \int_{\mathbb{R}^3} Q^{\epsilon+}(g, h^{n-1})(v) \text{sgn}(h^n) \langle v \rangle^l dv \\ & \leq \int_{\mathbb{R}^6} \int_{SS^2} b^\epsilon(\cos \theta) |v - v_*|^\gamma g_* |h^{n-1}| \langle v' \rangle^l dv dv_* d\sigma \\ & \leq (1 + \eta)^{\frac{\gamma}{2}} A^\epsilon \|g\|_{L^1} \|h^{n-1}\|_{L_{l+\gamma}^1} \\ & \quad + (1 + \frac{1}{\eta})^{\frac{\gamma}{2}} A^\epsilon \|g\|_{L_{l+\gamma}^1} \|h^{n-1}\|_{L^1} \\ & \quad + C(l, \gamma, \eta) A^\epsilon \|g\|_{L_l^1} \|h^{n-1}\|_{L_l^1}. \end{aligned}$$

Similar as in (3.9)

$$(3.23) \quad \begin{aligned} \int_{\mathbb{R}^3} Q^{\epsilon-}(g, h^n)(v) \text{sgn}(h^n) \langle v \rangle^l dv &= \int_{\mathbb{R}^3} b^\epsilon |v - v_*|^\gamma g_* |h^n| \langle v \rangle^l dv dv_* d\sigma \\ &\geq \frac{A^\epsilon}{(1 + \eta)^{\gamma/2}} \|g\|_{L^1} \|h^n\|_{L_{l+\gamma}^1} \\ &\quad - \eta^{-\gamma/2} A^\epsilon \|g\|_{L_\gamma^1} \|h^n\|_{L_l^1}. \end{aligned}$$

For the inner product $\langle Q_L(g, h^n), \text{sgn}(h^n) \langle v \rangle^l \rangle$, we can approximate Landau operator by Boltzmann operators. Let $b_\lambda(\cos \theta) = \lambda^{2s-2} b(\cos \theta) \mathbf{1}_{\theta \leq \lambda}$ for each $\lambda \leq \frac{\pi}{2}$, such that

$$\lim_{\lambda \downarrow 0} \int_{SS^2} b_\lambda(\cos \theta) \sin^2 \theta d\sigma = \Lambda.$$

Let Q_λ be the Boltzmann operator associated to the kernel $b_\lambda(\cos \theta) |v - v_*|^\gamma$, then by lemma 7.1 in [9], there holds

$$(3.24) \quad |\langle Q_L(g, h^n), \text{sgn}(h^n) \langle v \rangle^l \rangle_v - \langle Q_\lambda(g, h^n), \text{sgn}(h^n) \langle v \rangle^l \rangle_v| \lesssim \lambda \|g\|_{H_{l+\gamma+12}^3} \|h^n\|_{H_{l+\gamma+10}^5}.$$

By the uniform estimate (3.20) and our assumption on g and f_0 , we have

$$(3.25) \quad \begin{aligned} & \sup_{0 \leq t \leq T} \sup_n \|h^n(t)\|_{H_{l+\gamma+10}^5} \\ & \leq C(\|f_0\|_{L_{l+5\gamma/2+16}^1}, \|f_0\|_{H_{l+3\gamma/2+10}^5}, t, \sup_{0 \leq s \leq t} \|g(s)\|_{H_{2l+3\gamma+22}^5}, \sup_{0 \leq s \leq t} \|g(s)\|_{L_{l+5\gamma/2+16}^1}). \end{aligned}$$

Thanks to proposition 3.1, for $l \geq 4$, we derive that

$$\begin{aligned}
& \langle Q_\lambda(g, h^n), \text{sgn}(h^n) \langle v \rangle^l \rangle_v \\
&= \int_{\mathbb{R}^6} \int_{SS^2} b_\lambda |v - v_*|^\gamma g_* h^n (\text{sgn}(h^n(v')) \langle v' \rangle^l - \text{sgn}(h^n(v)) \langle v \rangle^l) dv dv_* d\sigma \\
&\leq \int_{\mathbb{R}^6} \int_{SS^2} b_\lambda |v - v_*|^\gamma g_* |h^n| (\langle v' \rangle^l - \langle v \rangle^l) dv dv_* d\sigma \\
&\leq - \int_{\mathbb{R}^6} \int_{SS^2} b_\lambda |v - v_*|^\gamma g_* |h^n| \langle v \rangle^l (1 - \cos^l \frac{\theta}{2}) dv dv_* d\sigma \\
&\quad + \int_{\mathbb{R}^6} \int_{SS^2} b_\lambda |v - v_*|^\gamma g_* |h^n| \langle v_* \rangle^l \sin^l \frac{\theta}{2} dv dv_* d\sigma \\
&\quad + C(l) \int_{\mathbb{R}^6} \int_{SS^2} b_\lambda |v - v_*|^\gamma g_* |h^n| \langle v \rangle^{l-2} \langle v_* \rangle^{l-2} \sin^2 \frac{\theta}{2} dv dv_* d\sigma.
\end{aligned}$$

For λ small enough, we have

$$\frac{\Lambda}{2} \leq \int_{SS^2} b_\lambda(\cos \theta) \sin^2 \theta d\sigma \leq 2\Lambda.$$

Thus we have

$$\int_{SS^2} b_\lambda(\cos \theta) (1 - \cos^l \frac{\theta}{2}) d\sigma \geq \int_{SS^2} b_\lambda(\cos \theta) \sin^2 \frac{\theta}{2} d\sigma \geq \frac{\Lambda}{8},$$

and

$$\int_{SS^2} b_\lambda(\cos \theta) \sin^2 \frac{\theta}{2} d\sigma \leq \int_{SS^2} b_\lambda(\cos \theta) \frac{\sin^2 \theta}{2} d\sigma \leq \Lambda,$$

and finally

$$\int_{SS^2} b_\lambda(\cos \theta) \sin^l \frac{\theta}{2} d\sigma \leq \frac{\lambda^2}{4} \int_{SS^2} b_\lambda(\cos \theta) \frac{\sin^2 \theta}{2} d\sigma \leq \frac{\lambda^2}{4} \Lambda.$$

With the help of the above three inequalities, we arrive at

$$\begin{aligned}
\langle Q_\lambda(g, h^n), \text{sgn}(h^n) \langle v \rangle^l \rangle_v &\leq -\frac{\Lambda}{16} \|g\|_{L^1} \|h^n\|_{L_{i+\gamma}^1} + \frac{\Lambda}{8} \|g\|_{L_\gamma^1} \|h^n\|_{L_i^1} \\
&\quad + C(l)\Lambda \|g\|_{L_i^1} \|h^n\|_{L_i^1} + \frac{\lambda^2}{4} \Lambda \|g\|_{L_{i+\gamma}^1} \|h^n\|_{L_\gamma^1}.
\end{aligned}$$

Let λ tend to 0, by (3.24) and the uniform estimate (3.25), we have

$$\begin{aligned}
(3.26) \quad \langle Q_L(g, h^n), \text{sgn}(h^n) \langle v \rangle^l \rangle_v &\leq -\frac{\Lambda}{16} \|g\|_{L^1} \|h^n\|_{L_{i+\gamma}^1} + \frac{\Lambda}{8} \|g\|_{L_\gamma^1} \|h^n\|_{L_i^1} \\
&\quad + C(l)\Lambda \|g\|_{L_i^1} \|h^n\|_{L_i^1}.
\end{aligned}$$

Choose η small enough such that

$$(1 + \eta)^{\frac{\gamma}{2}} A^\epsilon \leq \frac{A^\epsilon}{(1 + \eta)^{\gamma/2}} + \frac{1}{32} \epsilon^{2-2s} \Lambda,$$

and denote $a = \frac{A^\epsilon}{(1+\eta)^{\gamma/2}} + \frac{1}{32} \epsilon^{2-2s} \Lambda$, $\delta = \frac{1}{32} \epsilon^{2-2s} \Lambda$. Patch altogether (3.22), (3.23) and (3.26), we obtain

$$\begin{aligned}
\frac{d}{dt} \|h^n\|_{L_i^1} &\leq -(a + \delta) \|g\|_{L^1} \|h^n\|_{L_{i+\gamma}^1} + a \|g\|_{L^1} \|h^{n-1}\|_{L_{i+\gamma}^1} \\
&\quad + (1 + \frac{1}{\eta})^{\frac{\gamma}{2}} A^\epsilon \|g\|_{L_{i+\gamma}^1} \|h^{n-1}\|_{L^1} + C(l, \gamma, \eta) A^\epsilon \|g\|_{L_i^1} \|h^{n-1}\|_{L_i^1} \\
&\quad + \eta^{-\gamma/2} A^\epsilon \|g\|_{L_\gamma^1} \|h^n\|_{L_i^1} + \frac{\Lambda}{8} \epsilon^{2-2s} \|g\|_{L_\gamma^1} \|h^n\|_{L_i^1} \\
&\quad + C(l) \epsilon^{2-2s} \Lambda \|g\|_{L_i^1} \|h^n\|_{L_i^1}
\end{aligned}$$

For ease of notation, denote $K_1 = (1 + \frac{1}{\eta})^{\frac{\gamma}{2}} A^\epsilon$, $K_2 = C(l, \gamma, \eta) A^\epsilon \sup_{0 \leq s \leq t} \|g(s)\|_{L_l^1}$ and $K_3 = (\eta^{-\gamma/2} A^\epsilon + \frac{\Lambda}{8} \epsilon^{2-2s} + C(l) \epsilon^{2-2s} \Lambda) \sup_{0 \leq s \leq t} \|g(s)\|_{L_l^1}$. Then we have a much neater inequality on the interval $[0, t]$,

$$(3.27) \quad \begin{aligned} & \frac{d}{dt} \|h^n\|_{L_l^1} + (a + \delta) \|g\|_{L_l^1} \|h^n\|_{L_{l+\gamma}^1} \\ & \leq a \|g\|_{L_l^1} \|h^{n-1}\|_{L_{l+\gamma}^1} + K_3 \|h^n\|_{L_l^1} + (K_1 \|g\|_{L_{l+\gamma}^1} + K_2) \|h^{n-1}\|_{L_l^1}. \end{aligned}$$

Using the same technique as in the previous step, by defining $y^n(t_n) = e^{-K_3 t_n} \|h^n(t_n)\|_{L_l^1}$ and $x^n(t_n) = \int_0^{t_n} e^{-K_3 s} \|g(s)\|_{L_l^1} \|h^n(s)\|_{L_{l+\gamma}^1} ds$, for $n \geq 1$ and $t_n \in [0, t]$. Then for $n \geq 2$, we derive that

$$y^n(t_n) + (a + \delta) x^n(t_n) \leq a x^{n-1}(t_n) + \int_0^{t_n} (K_1 \|g(s)\|_{L_{l+\gamma}^1} + K_2) y^{n-1}(s) ds,$$

where we have used the initial condition $h^n(0) = 0$. Now denote $S^n(s) = \sum_{i=0}^{n-1} (\frac{a}{a+\delta})^i y^{n-i}(s)$ for $n \geq 1$ and $s \in [0, t]$, by recursive derivation, we obtain

$$\begin{aligned} S^n(t_n) + (a + \delta) x^n(t_n) & \leq (\frac{a}{a+\delta})^{n-1} y^1(t_n) + (\frac{a}{a+\delta})^{n-2} a x^1(t_n) \\ & \quad + \int_0^t (K_1 \|g(t_{n-1})\|_{L_{l+\gamma}^1} + K_2) S^{n-1}(t_{n-1}) dt_{n-1}. \end{aligned}$$

By previous estimates (3.11), we have

$$\begin{aligned} \sup_{0 \leq t_n \leq t} \{y^1(t_n) + x^1(t_n)\} & \leq C(\|f_0\|_{L_{l+\gamma}^1}, t, \sup_{0 \leq s \leq t} \|g(s)\|_{L_l^1}, \int_0^t \|g(s)\|_{L_{l+\gamma}^1} ds) \\ & \stackrel{\text{def}}{=} C(t). \end{aligned}$$

For ease of notation, for $n \geq 1$, let us define

$$b^n(t) = ((\frac{a}{a+\delta})^{n-1} + (\frac{a}{a+\delta})^{n-2} a) C(t).$$

Thus, by further recursive derivation, for any $t_n \in [0, t]$, we obtain

$$\begin{aligned} & S^n(t_n) + (a + \delta) x^n(t_n) \\ & \leq b^n(t) + \int_0^{t_n} (K_1 \|g(t_{n-1})\|_{L_{l+\gamma}^1} + K_2) S^{n-1}(t_{n-1}) dt_{n-1} \\ & \leq \sum_{i=2}^n b^i(t) \frac{(\int_0^{t_n} (K_1 \|g(s)\|_{L_{l+\gamma}^1} + K_2) ds)^{n-i}}{(n-i)!} \\ & \quad + \int_0^{t_n} \int_0^{t_{n-1}} \cdots \int_0^{t_2} S^1(t_1) \prod_{i=1}^{n-1} (K_1 \|g(t_i)\|_{L_{l+\gamma}^1} + K_2) dt_{n-1} dt_{n-2} \cdots dt_1 \\ & \leq \sum_{i=1}^n b^i(t) \frac{(\int_0^{t_n} (K_1 \|g(s)\|_{L_{l+\gamma}^1} + K_2) ds)^{n-i}}{(n-i)!}, \end{aligned}$$

where we used the fact $S^1(t_1) \leq C(t) \leq (a + \delta + 1)C(t) = b^1(t)$. Note that $b^n(t)$ is a geometric sequence and $b^n(t) = b^1(t)(\frac{a}{a+\delta})^{n-1}$ for any $n \geq 1$, thus we have

$$\begin{aligned} S^n(t_n) + (a + \delta) x^n(t_n) & \leq b^1(t) \sum_{i=1}^n (\frac{a}{a+\delta})^{i-1} \frac{(\int_0^{t_n} (K_1 \|g(s)\|_{L_{l+\gamma}^1} + K_2) ds)^{n-i}}{(n-i)!} \\ & = b^1(t) (\frac{a}{a+\delta})^{n-1} \sum_{i=1}^n \frac{(\frac{a+\delta}{a} \int_0^{t_n} (K_1 \|g(s)\|_{L_{l+\gamma}^1} + K_2) ds)^{n-i}}{(n-i)!} \\ & \leq b^1(t) (\frac{a}{a+\delta})^{n-1} \exp(\frac{a+\delta}{a} \int_0^{t_n} (K_1 \|g(s)\|_{L_{l+\gamma}^1} + K_2) ds). \end{aligned}$$

By recalling the definitions of S^n and x^n , we arrive at

$$\begin{aligned} & \sup_{0 \leq s \leq t} \|h^n(s)\|_{L_t^1} + \int_0^t e^{K_3(t-s)} \|g(s)\|_{L^1} \|h^n(s)\|_{L_{l+\gamma}^1} ds \\ & \leq b^1(t) \left(\frac{a}{a+\delta}\right)^{n-1} \exp\left(\frac{a+\delta}{a} \int_0^t (K_1 \|g(s)\|_{L_{l+\gamma}^1} + K_2) ds + K_3 t\right). \end{aligned}$$

Since the series $\sum_n (\frac{a}{a+\delta})^{n-1}$ is finite, we conclude that $\{f^n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^\infty([0, t]; L_l^1) \cap L^1([0, t]; L_{l+\gamma}^1)$. Due to the arbitrariness of $t \in [0, T]$, there is a function $f \in L^\infty([0, T]; L_l^1) \cap L^1([0, T]; L_{l+\gamma}^1)$ such that

$$\lim_{n \rightarrow \infty} \left\{ \sup_{0 \leq s \leq T} \|f^n(s) - f(s)\|_{L_l^1} + \int_0^T \|f^n(s) - f(s)\|_{L_{l+\gamma}^1} ds \right\} = 0$$

It is obvious that f is the solution to (3.1). Thus the non-positivity of f is ensured by the non-positivity of f^n .

Step 3: (High Order Moments and Smoothness)

In this step, we prove the solution f constructed in the previous step actually lies in $L^\infty([0, T]; L_{l+5\gamma/2+16}^1 \cap H_{l+\gamma+10}^5) \cap L^1([0, T]; L_{l+7\gamma/2+16}^1)$. Let $q = l + 5\gamma/2 + 16$. By lemma 3.1 and inequality (3.8), we first have

$$\begin{aligned} \int_{\mathbb{R}^3} Q^\epsilon(g, f)(v) \langle v \rangle^q dv &= \int_{\mathbb{R}^6} \int_{S^2} b^\epsilon(\cos \theta) |v - v_*|^\gamma g_* f(\langle v' \rangle^q - \langle v \rangle^q) dv dv_* d\sigma \\ &\leq -\frac{C_\epsilon}{2} \|g\|_{L^1} \|f\|_{L_{q+\gamma}^1} + C_\epsilon \|g\|_{L_\gamma^1} \|f\|_{L_q^1} + A_2 \|g\|_{L_{q+\gamma}^1} \|f\|_{L^1} + A_2 2^l \|g\|_{L_q^1} \|f\|_{L_q^1}. \end{aligned}$$

Next, according to [6], one has

$$\langle Q_L(g, h), \langle v \rangle^q \rangle \leq -\Lambda q \|g\|_{L^1} \|f\|_{L_{l+\gamma}^1} + \Lambda(4q+2)q \|g\|_{L_4^1} \|f\|_{L_q^1}.$$

Therefore we have

$$\frac{d}{dt} \|f\|_{L_q^1} + \frac{C_\epsilon}{2} \|g\|_{L^1} \|f\|_{L_{q+\gamma}^1} \leq C(M, \Lambda, q) \|g\|_{L_q^1} \|f\|_{L_q^1} + A_2 \|f_0\|_{L^1} \|g\|_{L_{q+\gamma}^1}.$$

By Gronwall's inequality, it is not difficult to derive

$$\sup_{0 \leq s \leq T} \|f\|_{L_q^1} + \int_0^T \|f(t)\|_{L_{q+\gamma}^1} dt \leq C(\|f_0\|_{L_q^1}, \sup_{0 \leq t \leq T} \|f\|_{L_q^1}, \int_0^T \|f(t)\|_{L_{q+\gamma}^1} dt).$$

Recalling the uniform estimate (3.25) and the convergence of $\{f^n\}_{n \in \mathbb{N}}$ in $L^\infty([0, T]; L_l^1)$, we also have $f \in L^\infty([0, T]; H_{l+\gamma+10}^5)$.

Step 4: (Uniqueness)

Suppose $f^1, f^2 \in L^\infty([0, T]; L_{l+5\gamma/2+16}^1 \cap H_{\gamma+10+l}^5) \cap L^1([0, T]; L_{l+7\gamma/2+16}^1)$ are two non-negative solutions of equation (3.1), set $h = f^1 - f^2$. Then h is a solution to the following equation,

$$(3.28) \quad \begin{cases} \partial_t h = Q^\epsilon(g, h) + \epsilon^{2-2s} Q_L(g, h) \\ h|_{t=0} = 0. \end{cases}$$

Observe that the above equation is as the same as the equation (3.21) if $h^{n-1} = h^n$. With the same argument until inequality (3.27), we have

$$\frac{d}{dt} \|h\|_{L_l^1} + C_1 \|h^n\|_{L_{l+\gamma}^1} \leq C_2 \|h\|_{L_l^1},$$

where C_1 and C_2 are some positive constants depending on M and m . Then we have

$$\|h(t)\|_{L_l^1} \leq \|h(0)\|_{L_l^1} e^{C_2 t},$$

which gives the uniqueness. \square

3.2. First result on the well-posedness of approximate equation (1.7). Based on the Picard iteration scheme, we derive that

Lemma 3.3. *Let $l \geq 4$ be a real number and N be an nonnegative integer. Let w_H, w_L, w be functions defined by*

$$(3.29) \quad w_H(N, l) = \max\{w(N, l) + 3\gamma/2 + 4, 2l + 3 + \gamma/2\},$$

$$(3.30) \quad w_L(N, l) = \max\{q(2, w(N, l) + \gamma + 4), q(N, 2l + 3), q(N + 1, l + \gamma/2 + 2)\} + \gamma,$$

$$(3.31) \quad w(N, l) = \frac{(N + s + 2)(2l + 3) - (N + 2)(l + \gamma/2)}{s}.$$

Suppose the non-negative datum $f_0 \in H_{w_H(N, l)}^{(N+2) \vee 3} \cap L_{w_L(N, l)}^1$ with $\|f_0\|_{L^1} > 0$, then our approximate equation (1.7) admits a non-negative solution f in $L^\infty([0, T^*]; H_l^N \cap L_{w(N, l)}^1)$ for some $T^* > 0$. Moreover, if $N \geq 2$ and $l \geq 8 + \gamma$, the solution is unique.

Proof. Consider the sequence of functions $\{f^n\}_{n \in \mathbb{N}}$ defined by

$$(3.32) \quad \begin{cases} f^0(t) = f_0, \text{ for any } t \geq 0; \\ \partial_t f^n = Q^\epsilon(f^{n-1}, f^n) + \epsilon^{2-2s} Q_L(f^{n-1}, f^n), \quad n \geq 1, \\ f^n|_{t=0} = f_0. \end{cases}$$

We first mention that equation (3.32) conserves mass, that is, $\|f^n(t)\|_{L^1} = \|f_0\|_{L^1}$ for any $n \geq 0$ and $t \geq 0$. By previous lemma, $f^n \geq 0$ for any $n \in \mathbb{N}$.

Step 1: (Uniform L_l^1 Upper Bound)

In this step we prove that $\{f^n\}_n$ has uniform upper bound in $L^\infty([0, T^*(l)]; L_l^1)$ with respect to n for some $T^*(l) > 0$ if $f_0 \in L_{l+\gamma}^1$. Thanks to proposition 3.1, for any $l \geq 4$, we have

$$\begin{aligned} \langle Q^\epsilon(f^{n-1}, f^n), \langle v \rangle^l \rangle &= \int_{\mathbb{R}^6} \int_{SS^2} b^\epsilon |v - v_*|^\gamma f_*^{n-1} f^n (\langle v \rangle^l - \langle v \rangle^l) dv dv_* d\sigma \\ &\leq - \int_{\mathbb{R}^6} \int_{SS^2} b^\epsilon |v - v_*|^\gamma f_*^{n-1} f^n \langle v \rangle^l (1 - \cos^l \frac{\theta}{2}) dv dv_* d\sigma \\ &\quad + \int_{\mathbb{R}^6} \int_{SS^2} b^\epsilon |v - v_*|^\gamma f_*^{n-1} f^n \langle v_* \rangle^l \sin^l \frac{\theta}{2} dv dv_* d\sigma \\ &\quad + C(l) \int_{\mathbb{R}^6} \int_{SS^2} b^\epsilon |v - v_*|^\gamma f_*^{n-1} f^n \langle v \rangle^{l-2} \langle v_* \rangle^{l-2} \sin^2 \frac{\theta}{2} dv dv_* d\sigma. \end{aligned}$$

In the following, denote $A_2^\epsilon = \int_{SS^2} b^\epsilon \sin^2 \frac{\theta}{2} d\sigma \leq \frac{A_2}{2}$, then we have

$$\int_{SS^2} b^\epsilon (\cos \theta) (1 - \cos^l \frac{\theta}{2}) d\sigma \geq \frac{3}{2} A_2^\epsilon,$$

and

$$\int_{SS^2} b^\epsilon (\cos \theta) \sin^l \frac{\theta}{2} d\sigma \leq \frac{1}{2} A_2^\epsilon,$$

where we used $1 - \cos^l \frac{\theta}{2} \geq 1 - \cos^4 \frac{\theta}{2} \geq \frac{3}{2} \sin^2 \frac{\theta}{2}$ and $\sin^l \frac{\theta}{2} \leq \frac{1}{2} \sin^2 \frac{\theta}{2}$. Together with $|v - v_*|^\gamma \geq \frac{3}{4} \langle v \rangle^\gamma - c_1 \langle v_* \rangle^\gamma$, $|v - v_*|^\gamma \leq 2(\langle v \rangle^\gamma + \langle v_* \rangle^\gamma)$, and $|v - v_*|^\gamma \leq \langle v \rangle^\gamma \langle v_* \rangle^\gamma$, we obtain

$$(3.33) \quad \begin{aligned} \langle Q^\epsilon(f^{n-1}, f^n), \langle v \rangle^l \rangle &\leq -\frac{9}{8} A_2^\epsilon \|f^{n-1}\|_{L^1} \|f^n\|_{L_{l+\gamma}^1} + \frac{3}{2} c_1 A_2^\epsilon \|f^{n-1}\|_{L_l^1} \|f^n\|_{L_l^1} \\ &\quad + A_2^\epsilon \|f^{n-1}\|_{L_{l+\gamma}^1} \|f^n\|_{L^1} + A_2^\epsilon \|f^{n-1}\|_{L_l^1} \|f^n\|_{L_l^1} \\ &\quad + C(l) A_2^\epsilon \|f^{n-1}\|_{L_l^1} \|f^n\|_{L_l^1}. \end{aligned}$$

Recalling (3.10), we have

$$(3.34) \quad \langle Q_L(f^{n-1}, f^n), \langle v \rangle^l \rangle \leq -l\Lambda \|f^{n-1}\|_{L^1} \|f^n\|_{L_{l+\gamma}^1} + (4l + 2)l\Lambda \|f^{n-1}\|_{L_{l+4}^1} \|f^n\|_{L_l^1}.$$

With (3.33) and (3.34) in hand, we have

$$(3.35) \quad \begin{aligned} & \frac{d}{dt} \|f^n\|_{L_l^1} + \frac{9}{8} A_2^\epsilon \|f_0\|_{L^1} \|f^n\|_{L_{l+\gamma}^1} \\ & \leq A_2^\epsilon \|f_0\|_{L^1} \|f^{n-1}\|_{L_{l+\gamma}^1} + C(\epsilon, l, \Lambda) \|f^{n-1}\|_{L_l^1} \|f^n\|_{L_l^1}, \end{aligned}$$

where we denote $C(\epsilon, l, \Lambda) = A_2^\epsilon + \frac{3}{2} c_1 A_2^\epsilon + C(l) A_2^\epsilon + \epsilon^{2-2s} (4l+2) l \Lambda$. For simplicity, denote $m(l) = \|f_0\|_{L_l^1}$. For any $n \in \mathbb{N}$ and $l \geq 0$, define

$$T^*(l) = \frac{\min\{\log\{\frac{11C(\epsilon, l, \Lambda)m^2(l)}{A_2^\epsilon m(0)m(l+\gamma)} + 1\}, \log(10/9)\}}{11C(\epsilon, l, \Lambda)m(l)},$$

and

$$C_{n,l} = \sup_{0 \leq t \leq T^*(l)} \sup_{0 \leq k \leq n} \|f^k(t)\|_{L_l^1}.$$

We claim that for any $n \in \mathbb{N}$,

$$(3.36) \quad C_{n,l} \leq 11m(l).$$

We will prove (3.36) by induction. First, it is obvious $C_{0,l} \leq 11m(l)$. Next, fix a $n \geq 1$, suppose $C_{n-1,l} \leq 11m(l)$, then on the interval $[0, T^*(l)]$, for any $1 \leq k \leq n$, from (3.35), we have

$$(3.37) \quad \begin{aligned} & \frac{d}{dt} \|f^k\|_{L_l^1} + \frac{9}{8} A_2^\epsilon m(0) \|f^k\|_{L_{l+\gamma}^1} \\ & \leq A_2^\epsilon m(0) \|f^{k-1}\|_{L_{l+\gamma}^1} + 11C(\epsilon, l, \Lambda) m(l) \|f^k\|_{L_l^1}, \end{aligned}$$

Thus for any $t \in [0, T^*(l)]$ and $1 \leq k \leq n$, we derive that

$$\begin{aligned} & e^{-11C(\epsilon, l, \Lambda) m(l) t} \|f^k(t)\|_{L_l^1} + \frac{9}{8} A_2^\epsilon m(0) \int_0^t e^{-11C(\epsilon, l, \Lambda) m(l) s} \|f^k(s)\|_{L_{l+\gamma}^1} ds \\ & \leq A_2^\epsilon m(0) \int_0^t e^{-11C(\epsilon, l, \Lambda) m(l) s} \|f^{k-1}(s)\|_{L_{l+\gamma}^1} ds + m(l). \end{aligned}$$

Multiplying the above inequality by $(\frac{8}{9})^{n-k}$ and taking sum over $1 \leq k \leq n$, we obtain

$$\begin{aligned} & e^{-11C(\epsilon, l, \Lambda) m(l) t} \sum_{k=1}^n \left(\frac{8}{9}\right)^{n-k} \|f^k(t)\|_{L_l^1} + \frac{9}{8} A_2^\epsilon m(0) \int_0^t e^{-11C(\epsilon, l, \Lambda) m(l) s} \|f^n(s)\|_{L_{l+\gamma}^1} ds \\ & \leq \left(\frac{8}{9}\right)^{n-1} A_2^\epsilon m(0) m(l+\gamma) \frac{1 - e^{-11C(\epsilon, l, \Lambda) m(l) t}}{11C(\epsilon, l, \Lambda) m(l)} + m(l) \sum_{k=1}^n \left(\frac{8}{9}\right)^{n-k}. \end{aligned}$$

Observing that $\sum_{k=1}^n (\frac{8}{9})^{n-k} \leq 9$, we arrive at

$$\begin{aligned} & \sum_{k=1}^n \left(\frac{8}{9}\right)^{n-k} \|f^k(t)\|_{L_l^1} + \frac{9}{8} A_2^\epsilon m(0) \int_0^t e^{11C(\epsilon, l, \Lambda) m(l) (t-s)} \|f^n(s)\|_{L_{l+\gamma}^1} ds \\ & \leq A_2^\epsilon m(0) m(l+\gamma) \|f_0\|_{L_{l+\gamma}^1} \frac{e^{11C(\epsilon, l, \Lambda) m(l) t} - 1}{11C(\epsilon, l, \Lambda) m(l)} + 9m(l) e^{11C(\epsilon, l, \Lambda) m(l) t}. \end{aligned}$$

Thus we have

$$\begin{aligned} \sup_{0 \leq t \leq T^*(l)} \|f^n(t)\|_{L_l^1} & \leq A_2^\epsilon m(0) \|f_0\|_{L_{l+\gamma}^1} \frac{e^{11C(\epsilon, l, \Lambda) m(l) T^*(l)} - 1}{11C(\epsilon, l, \Lambda) m(l)} + 9\|f_0\|_{L_l^1} e^{11C(\epsilon, l, \Lambda) m(l) T^*(l)} \\ & \leq 11m(l), \end{aligned}$$

by the definition of $T^*(l)$. That is, $C_n \leq 11m(l)$. Therefore the claim (3.36) is proved, which implies

$$(3.38) \quad \sup_{0 \leq t \leq T^*(l)} \sup_{n \geq 0} \|f^n(t)\|_{L_l^1} \leq 11m(l).$$

Step 2: (Uniform H_l^N Upper Bound)

In this step, we shall use the energy estimate to get the uniform upper bound of L_l^N norm of f^n with respect to n . Fix an α with $|\alpha| \leq N$, one has

$$\partial_t \partial_v^\alpha f^n = \sum_{\alpha_1 + \alpha_2 = \alpha} \binom{\alpha}{\alpha_1} [Q^\epsilon(\partial_v^{\alpha_1} f^{n-1}, \partial_v^{\alpha_2} f^n) + \epsilon^{2-2s} Q_L(\partial_v^{\alpha_1} f^{n-1}, \partial_v^{\alpha_2} f^n)].$$

As before, we have

$$\begin{aligned} & \langle M^\epsilon(f^{n-1}, \partial_v^\alpha f^n), \partial_v^\alpha f^n \langle v \rangle^{2l} \rangle \\ = & \langle M^\epsilon(f^{n-1}, \partial_v^\alpha f^n \langle v \rangle^l), \partial_v^\alpha f^n \langle v \rangle^l \rangle \\ & + \{ \langle M^\epsilon(f^{n-1}, \partial_v^\alpha f^n) \langle v \rangle^l - M^\epsilon(f^{n-1}, \partial_v^\alpha f^n \langle v \rangle^l), \partial_v^\alpha f^n \langle v \rangle^l \} \end{aligned}$$

By coercivity estimate (2.5) and commutator estimates (2.7), (2.8), we have

$$\begin{aligned} (3.39) \quad & \langle M^\epsilon(f^{n-1}, \partial_v^\alpha f^n), \partial_v^\alpha f^n \langle v \rangle^{2l} \rangle + C_1(f_0) \|\partial_v^\alpha f^n\|_{\epsilon, l+\gamma/2}^2 \\ & \lesssim C_2(f_0) \|\partial_v^\alpha f^n\|_{L_{l+\gamma/2}^2}^2 + \|f^{n-1}\|_{L_{2l+1}^1} \|\partial_v^\alpha f^n\|_{\epsilon, l+\gamma/2} \|\partial_v^\alpha f^n\|_{L_{l+\gamma/2}^2}. \end{aligned}$$

By upper bound estimate (2.1) and commutator estimates (2.7), (2.8), for $|\alpha_2| \leq N-1$, we have

$$\begin{aligned} (3.40) \quad & \langle M^\epsilon(\partial_v^{\alpha_1} f^{n-1}, \partial_v^{\alpha_2} f^n), \partial_v^\alpha f^n \langle v \rangle^{2l} \rangle \\ = & \langle M^\epsilon(\partial_v^{\alpha_1} f^{n-1}, \partial_v^{\alpha_2} f^n \langle v \rangle^l), \partial_v^\alpha f^n \langle v \rangle^l \rangle \\ & + \{ \langle M^\epsilon(\partial_v^{\alpha_1} f^{n-1}, \partial_v^{\alpha_2} f^n) \langle v \rangle^l, \partial_v^\alpha f^n \langle v \rangle^l \} \\ & - \langle M^\epsilon(\partial_v^{\alpha_1} f^{n-1}, \partial_v^{\alpha_2} f^n \langle v \rangle^l), \partial_v^\alpha f^n \langle v \rangle^l \rangle \\ \lesssim & \|\partial_v^{\alpha_1} f^{n-1}\|_{L_{\gamma+2}^1} \|\partial_v^{\alpha_2} f^n\|_{H_{l+\gamma/2+2}^s} \|\partial_v^\alpha f^n\|_{H_{l+\gamma/2}^s} \\ & + \epsilon^{2-2s} \|\partial_v^{\alpha_1} f^{n-1}\|_{L_{\gamma+2}^1} \|\partial_v^{\alpha_2} f^n\|_{H_{l+\gamma/2+2}^1} \|\partial_v^\alpha f^n\|_{H_{l+\gamma/2}^1} \\ & + \|\partial_v^{\alpha_1} f^{n-1}\|_{L_{2l+1}^1} \|\partial_v^{\alpha_2} f^n\|_{H_{l+\gamma/2}^s} \|\partial_v^\alpha f^n\|_{L_{l+\gamma/2}^2} \\ & + \epsilon^{2-2s} \|\partial_v^{\alpha_1} f^{n-1}\|_{L_{\gamma+3}^1} \|\partial_v^{\alpha_2} f^n\|_{H_{l+\gamma/2}^1} \|\partial_v^\alpha f^n\|_{L_{l+\gamma/2}^2}. \end{aligned}$$

When $N = 0$, by (3.39), we have

$$\begin{aligned} & \langle M^\epsilon(f^{n-1}, f^n), f^n \langle v \rangle^{2l} \rangle + \frac{C_1(f_0)}{2} \|f^n\|_{\epsilon, l+\gamma/2}^2 \\ \lesssim & C_2(f_0) \|f^n\|_{L_{l+\gamma/2}^2}^2 + \frac{1}{C_1(f_0)} \|f^{n-1}\|_{L_{2l+1}^1}^2 \|f^n\|_{L_{l+\gamma/2}^2}^2. \end{aligned}$$

By (3.38), there holds

$$\sup_{0 \leq t \leq T^*(2l+1)} \sup_{n \geq 0} \|f^n(t)\|_{L_{2l+1}^1} \leq 11m(2l+1),$$

so we have

$$\frac{d}{dt} \|f^n\|_{L_l^2}^2 + C_1(f_0) \|f^n\|_{\epsilon, l+\gamma/2}^2 \lesssim C(\|f_0\|_{L_{2l+1}^1}, \|f_0\|_{L \log L}) \|f^n\|_{L_{l+\gamma/2}^2}^2.$$

Thanks to the fact

$$\|f^n\|_{L_{l+\gamma/2}^2}^2 \leq \eta \|f^n\|_{H_{l+\gamma/2}^s}^2 + C(\eta) \|f^n\|_{L_{l+\gamma/2}^1}^2,$$

we have

$$\frac{d}{dt} \|f^n\|_{L_l^2}^2 + \frac{C_1(f_0)}{2} \|f^n\|_{\epsilon, l+\gamma/2}^2 \leq C(\|f_0\|_{L_{2l+1}^1}, \|f_0\|_{L \log L}).$$

By Gronwall's inequality, we obtain

$$\sup_{0 \leq t \leq T^*(2l+1)} \|f^n(t)\|_{L_l^2}^2 + \int_0^{T^*(2l+1)} \|f^n(s)\|_{\epsilon, l+\gamma/2}^2 ds \leq C(\|f_0\|_{L_{2l+1}^1}, \|f_0\|_{L_l^2}).$$

With the help of uniform L_l^2 norm and the above inequality, we can prove in a similar manner as in the second step in the proof of theorem 1.1,

$$\sup_{0 \leq t \leq T^*(\phi(s, l))} \|f^n(t)\|_{H_l^s} \leq C(\|f_0\|_{L_{\phi(s, l)}^1}, \|f_0\|_{H_l^s}).$$

where $\phi(s, l) = \frac{(2l+4)(2+s)-2l}{s}$.

Now we turn to higher order regularity. Taking into account the fact $W^\epsilon(\xi) \leq \langle \xi \rangle$, for the fixed ϵ , by (3.39) and (3.40), we have

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|f^n\|_{H_l^1}^2 \right) + \frac{C_1(f_0)}{2} \epsilon^{2-2s} \|f^n\|_{H_{l+\gamma/2}^2}^2 & \lesssim \|f^n\|_{H_{l+\gamma/2}^1}^2 + \|f^{n-1}\|_{H_{2l+3}^1} \|f^n\|_{H_{l+\gamma/2}^1}^2 \\ & + \|f^{n-1}\|_{H_6^1} \|f^n\|_{H_{l+2+\gamma/2}^1} \|f^n\|_{H_{l+\gamma/2}^2}. \end{aligned}$$

Thanks interpolation theory and Young's inequality, one has

$$\begin{aligned}
\|f^n\|_{H_{l+\gamma/2}^1}^2 &\leq \|f^n\|_{H_{l+\gamma/2}^2} \|f^n\|_{L_{l+\gamma/2}^2} \\
&\leq \frac{C_1(f_0)}{8} \epsilon^{2-2s} \|f^n\|_{H_{l+\gamma/2}^2}^2 + \frac{2}{C_1(f_0) \epsilon^{2-2s}} \|f^n\|_{L_{l+\gamma/2}^2}^2, \\
\|f^{n-1}\|_{H_{2l+3}^1} \|f^n\|_{H_{l+\gamma/2}^1}^2 &\leq \|f^{n-1}\|_{H_{l+\gamma/2}^2}^{1/2} \|f^{n-1}\|_{L_{3l+6-\gamma/2}^2}^{1/2} \|f^n\|_{H_{l+\gamma/2}^2} \|f^n\|_{L_{l+\gamma/2}^2} \\
&\leq \frac{C_1(f_0)}{8} \epsilon^{2-2s} \|f^n\|_{H_{l+\gamma/2}^2}^2 \\
&\quad + \frac{2}{C_1(f_0) \epsilon^{2-2s}} \|f^{n-1}\|_{H_{l+\gamma/2}^2} \|f^{n-1}\|_{L_{3l+6-\gamma/2}^2} \|f^n\|_{L_{l+\gamma/2}^2}^2 \\
&\leq \frac{C_1(f_0)}{8} \epsilon^{2-2s} \|f^n\|_{H_{l+\gamma/2}^2}^2 + \frac{C_1(f_0)}{32} \epsilon^{2-2s} \|f^{n-1}\|_{H_{l+\gamma/2}^2}^2 \\
&\quad + \frac{32}{(C_1(f_0) \epsilon^{2-2s})^3} \|f^{n-1}\|_{L_{3l+6-\gamma/2}^2}^2 \|f^n\|_{L_{l+\gamma/2}^2}^4,
\end{aligned}$$

and finally

$$\begin{aligned}
&\|f^{n-1}\|_{H_6^1} \|f^n\|_{H_{l+2+\gamma/2}^1} \|f^n\|_{H_{l+\gamma/2}^2} \\
&\leq \|f^{n-1}\|_{H_6^2}^{\frac{1-s}{2}} \|f^{n-1}\|_{H_6^2}^{\frac{1}{2-s}} \|f^n\|_{H_{l+\gamma/2}^2}^{3/2} \|f^n\|_{L_{l+4+\gamma/2}^2}^{1/2} \\
&\leq \frac{C_1(f_0)}{8} \epsilon^{2-2s} \|f^n\|_{H_{l+\gamma/2}^2}^2 \\
&\quad + \frac{1}{4} \left(\frac{32}{3C_1(f_0)} \epsilon^{2-2s} \right)^3 \|f^{n-1}\|_{H_6^2}^{\frac{4(1-s)}{2-s}} \|f^{n-1}\|_{H_6^2}^{\frac{4}{2-s}} \|f^n\|_{L_{l+4+\gamma/2}^2}^2 \\
&\leq \frac{C_1(f_0)}{8} \epsilon^{2-2s} \|f^n\|_{H_{l+\gamma/2}^2}^2 + \frac{C_1(f_0)}{32} \epsilon^{2-2s} \|f^{n-1}\|_{H_6^2}^2 \\
&\quad + \frac{1}{p} (q\eta)^{-\frac{p}{q}} \left(\frac{1}{4} \left(\frac{32}{3C_1(f_0)} \epsilon^{2-2s} \right)^3 \right)^{\frac{2-s}{s}} \|f^{n-1}\|_{H_6^2}^{\frac{4}{s}} \|f^n\|_{L_{l+4+\gamma/2}^2}^{\frac{2(2-s)}{s}},
\end{aligned}$$

we have used the Young's inequality (1.10) with $p = \frac{2-s}{s}$, $q = \frac{2-s}{s(1-s)}$ and $\eta = \frac{C_1(f_0)}{32} \epsilon^{2-2s}$. Thus we arrive at for any $n \geq 1$,

$$\frac{d}{dt} \|f^n\|_{H_l^1}^2 + \frac{1}{4} C_1(f_0) \epsilon^{2-2s} \|f^n\|_{H_{l+\gamma/2}^2}^2 \leq \frac{1}{8} C_1(f_0) \epsilon^{2-2s} \|f^{n-1}\|_{H_{l+\gamma/2}^2}^2 + M,$$

where M is the uniform upper bound of $\|f^n\|_{H_6^2}$, $\|f^n\|_{L_{l+5}^2}$ and $\|f^n\|_{L_{3l+6}^1}$ with respect to n on the time interval $[0, T^*]$. Here $T^* = T^*(\max\{\phi(s, 6), 3l+6\})$. With the same technique as in dealing with (3.37), we obtain

$$\begin{aligned}
&\|f^n(t)\|_{H_l^1}^2 + \frac{1}{4} C_1(f_0) \epsilon^{2-2s} \int_0^t \|f^n(r)\|_{H_{l+\gamma/2}^2}^2 dr \\
&\leq \frac{1}{8} C_1(f_0) \epsilon^{2-2s} \|f_0\|_{H_{l+\gamma/2}^2}^2 t + 2(Mt + \|f_0\|_{H_l^1}^2).
\end{aligned}$$

The above inequality is true for any $n \geq 1$ and $t \in [0, T^*]$, so we have the desired result

$$\sup_n \sup_{0 \leq t \leq T^*} \|f^n(t)\|_{H_l^1} \leq C(\|f_0\|_{H_{l+\gamma/2}^2}, \|f_0\|_{L_{\max\{\phi(s, 6), 3l+6\}}^1}, T^*).$$

Continuing the argument, there will be a function $q : \mathbb{N} \times \mathbb{R}_+ \rightarrow \mathbb{R}$, such that

$$(3.41) \quad \sup_n \sup_{0 \leq t \leq T^*(q(N, l))} \|f^n(t)\|_{H_l^N} \leq C(\|f_0\|_{H_{l+\gamma/2}^{N+1}}, \|f_0\|_{L_{q(N, l)+\gamma}^1}).$$

Step 3: (Cauchy Sequence)

Now we are ready to prove $\{f^n\}_n$ is a Cauchy sequence in $L^\infty([0, T^*]; L_l^1)$. Set $h^n = f^{n+1} - f^n$ for $n \geq 0$. Then for $n \geq 1$, h^n is the solution to the following equation

$$(3.42) \quad \begin{cases} \partial_t h^n = M^\epsilon(f^n, h^n) + M^\epsilon(h^{n-1}, f^n), \\ h^n|_{t=0} = 0. \end{cases}$$

As the same as (3.33), we have

$$\begin{aligned}
 (3.43) \quad \langle Q^\epsilon(f^n, h^n), \operatorname{sgn}(h^n)\langle v \rangle^l \rangle &\leq \langle Q^\epsilon(f^n, |h^n|), \langle v \rangle^l \rangle \\
 &\leq -\frac{9}{8}A_2^\epsilon \|f^n\|_{L^1} \|h^n\|_{L_{l+\gamma}^1} + \frac{3}{2}A_2^\epsilon \|f^n\|_{L_\gamma^1} \|h^n\|_{L_l^1} \\
 &\quad + A_2^\epsilon \|f^n\|_{L_{l+\gamma}^1} \|h^n\|_{L^1} + A_2^\epsilon \|f^n\|_{L_l^1} \|h^n\|_{L_\gamma^1} \\
 &\quad + C(l)A_2^\epsilon \|f^n\|_{L_l^1} \|h^n\|_{L_l^1}.
 \end{aligned}$$

As the same as (3.26), we have

$$\begin{aligned}
 (3.44) \quad \langle Q_L(f^n, h^n), \operatorname{sgn}(h^n)\langle v \rangle^l \rangle_v &\leq -\frac{\Lambda}{16} \|f^n\|_{L^1} \|h^n\|_{L_{l+\gamma}^1} + \frac{\Lambda}{8} \|f^n\|_{L_\gamma^1} \|h^n\|_{L_l^1} \\
 &\quad + C(l)\Lambda \|f^n\|_{L_l^1} \|h^n\|_{L_l^1}.
 \end{aligned}$$

Applying proposition 3.1 again, we obtain

$$\begin{aligned}
 (3.45) \quad &\langle Q^\epsilon(h^{n-1}, f^n), \operatorname{sgn}(h^n)\langle v \rangle^l \rangle \\
 &\leq \int_{\mathbb{R}^6} \int_{SS^2} b^\epsilon |v - v_*|^\gamma |h_*^{n-1}| f^n(\langle v' \rangle^l + \langle v \rangle^l) dv dv_* d\sigma \\
 &\leq A_2^\epsilon (\|f^n\|_{L^1} \|h^{n-1}\|_{L_{l+\gamma}^1} + \|f^n\|_{L_\gamma^1} \|h^{n-1}\|_{L_l^1}) \\
 &\quad + C(l)A_2^\epsilon \|f^n\|_{L_l^1} \|h^{n-1}\|_{L_l^1} \\
 &\quad + A^\epsilon \|f^n\|_{L_{l+\gamma}^1} \|h^{n-1}\|_{L_\gamma^1}.
 \end{aligned}$$

Recalling the Landau operator Q_L can be rewritten as:

$$Q_L(g, h) = \sum_{i,j=1}^3 (a_{ij} * g) \partial_{ij} h - (c * g) h,$$

we have

$$\begin{aligned}
 (3.46) \quad \langle Q_L(h^{n-1}, f^n), \operatorname{sgn}(h^n)\langle v \rangle^l \rangle &= \sum_{i,j=1}^3 \langle (a_{ij} * h^{n-1}) \partial_{ij} f^n, \operatorname{sgn}(h^n)\langle v \rangle^l \rangle \\
 &\quad - \langle (c * h^{n-1}) f^n, \operatorname{sgn}(h^n)\langle v \rangle^l \rangle \\
 &\leq \Lambda \|h^{n-1}\|_{L_{\gamma+2}^1} \|f^n\|_{H_{l+\gamma+4}^2} \\
 &\quad + 2\Lambda(\gamma+3) \|h^{n-1}\|_{L_\gamma^1} \|f^n\|_{L_{l+\gamma}^1}.
 \end{aligned}$$

Patch all together inequalities (3.43), (3.44), (3.45) and (3.46), we obtain

$$\begin{aligned}
 (3.47) \quad &\frac{d}{dt} \|h^n\|_{L_l^1} + \frac{9}{8} A_2^\epsilon m(0) \|h^n\|_{L_{l+\gamma}^1} \\
 &\leq A_2^\epsilon m(0) \|h^{n-1}\|_{L_{l+\gamma}^1} + K_1 \|h^n\|_{L_l^1} + K_2 \|h^{n-1}\|_{L_l^1},
 \end{aligned}$$

where K_1 and K_2 are some constants depending at most on the uniform upper bound of $\|f^n\|_{H_{l+\gamma+4}^2}$, which is bounded by a constant depending on $\|f_0\|_{H_{l+3\gamma/2+4}^3}, \|f_0\|_{L_{q(2,l+\gamma+4)+\gamma}^1}$. With a similar argument as in the previous lemma, for any $t \in [0, T^*(q(2, l + \gamma + 4))]$, we can conclude

$$(3.48) \quad \|h^n(t)\|_{L_l^1} \leq \left(\frac{8}{9}\right)^n M(t) \exp\left(\frac{9K_2 t}{8} + K_1 t\right),$$

where $M(t) = \frac{9}{8} A_2^\epsilon m(0) \int_0^t e^{-K_1 s} \|h^0(s)\|_{L_{l+\gamma}^1} ds + 22m(l)$. Thus $\sum_n \|h^n(t)\|_{L_l^1}$ is finite and $\{f^n(t)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in L_l^1 . Due to the arbitrariness of $t \in [0, T^*(q(2, l + \gamma + 4))]$, there is a function $f \in L^\infty([0, T^*]; L_l^1)$ such that

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq T^*} \|f^n(t) - f(t)\|_{L_l^1} = 0$$

In the following, we prove $\{f^n(t)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in H_l^N . Fix an α with $|\alpha| \leq N$, one has

$$\partial_t \partial_v^\alpha h^n = \sum_{\alpha_1 + \alpha_2 = \alpha} \binom{\alpha}{\alpha_1} [M^\epsilon(\partial_v^{\alpha_1} f^n, \partial_v^{\alpha_2} h^n) + M^\epsilon(\partial_v^{\alpha_1} h^{n-1}, \partial_v^{\alpha_2} f^n)].$$

Then we have

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|\partial_v^\alpha h^n\|_{L_t^2}^2 \right) &= \sum_{\alpha_1 + \alpha_2 = \alpha} \binom{\alpha}{\alpha_1} [\langle M^\epsilon(\partial_v^{\alpha_1} f^n, \partial_v^{\alpha_2} h^n), \partial_v^\alpha h^n \langle v \rangle^{2l} \rangle \\ &\quad + \langle M^\epsilon(\partial_v^{\alpha_1} h^{n-1}, \partial_v^{\alpha_2} f^n), \partial_v^\alpha h^n \langle v \rangle^{2l} \rangle] \\ &\stackrel{\text{def}}{=} \sum_{\alpha_1 + \alpha_2 = \alpha} \binom{\alpha}{\alpha_1} [\mathfrak{I}_1(\alpha_1, \alpha_2) + \mathfrak{I}_2(\alpha_1, \alpha_2)]. \end{aligned}$$

As the same as (3.39), on the time interval $[0, T^*(2l+1)]$, we have

$$\begin{aligned} &\mathfrak{I}_1(0, \alpha) + C_1(f_0) \|\partial_v^\alpha h^n\|_{\epsilon, l+\gamma/2}^2 \\ &\lesssim C_2(f_0) \|\partial_v^\alpha h^n\|_{L_{l+\gamma/2}^2}^2 + \|f^n\|_{L_{2l+1}^1} \|\partial_v^\alpha h^n\|_{\epsilon, l+\gamma/2} \|\partial_v^\alpha h^n\|_{L_{l+\gamma/2}^2}, \end{aligned}$$

and thus

$$\mathfrak{I}_1(0, \alpha) + \frac{C_1(f_0)}{2} \|\partial_v^\alpha h^n\|_{\epsilon, l+\gamma/2}^2 \lesssim C(\|f_0\|_{L_{2l+1}^1}, \|f_0\|_{L^2}) \|\partial_v^\alpha h^n\|_{L_{l+\gamma/2}^2}.$$

As the same as (3.40), for $|\alpha_2| \leq |\alpha| - 1 \leq N-1$ and any $\eta > 0$, on the time interval $[0, T^*(q(N, 2l+3))]$, we have

$$\begin{aligned} \mathfrak{I}_1(\alpha_1, \alpha_2) &\lesssim \|\partial_v^{\alpha_1} f^n\|_{L_{\gamma+2}^1} \|\partial_v^{\alpha_2} h^n\|_{H_{l+\gamma/2+2}^s} \|\partial_v^\alpha h^n\|_{H_{l+\gamma/2}^s} \\ &\quad + \epsilon^{2-2s} \|\partial_v^{\alpha_1} f^n\|_{L_{\gamma+2}^1} \|\partial_v^{\alpha_2} h^n\|_{H_{l+\gamma/2+2}^1} \|\partial_v^\alpha h^n\|_{H_{l+\gamma/2}^1} \\ &\quad + \|\partial_v^{\alpha_1} f^n\|_{L_{2l+1}^1} \|\partial_v^{\alpha_2} h^n\|_{H_{l+\gamma/2}^s} \|\partial_v^\alpha h^n\|_{L_{l+\gamma/2}^2} \\ &\quad + \epsilon^{2-2s} \|\partial_v^{\alpha_1} f^n\|_{L_{\gamma+3}^1} \|\partial_v^{\alpha_2} h^n\|_{H_{l+\gamma/2}^1} \|\partial_v^\alpha h^n\|_{L_{l+\gamma/2}^2} \\ &\lesssim \|f^n\|_{H_{2l+3}^N} \|h^n\|_{H_{l+\gamma/2+2}^N} \|\partial_v^\alpha h^n\|_{\epsilon, l+\gamma/2}, \end{aligned}$$

which implies, for any $\eta > 0$,

$$\mathfrak{I}_1(\alpha_1, \alpha_2) - \eta \|\partial_v^\alpha h^n\|_{\epsilon, l+\gamma/2}^2 \lesssim C(\eta, \|f_0\|_{H_{2l+3+\gamma/2}^{N+1}}, \|f_0\|_{L_{q(N, 2l+3)+\gamma}^1}) \|h^n\|_{H_{l+\gamma/2+2}^N}^2.$$

Similarly, on the time interval $[0, T^*(q(N+1, l+\gamma/2+2))]$, we have

$$\mathfrak{I}_2(\alpha_1, \alpha_2) \lesssim \|h^{n-1}\|_{H_{2l+3}^N} \|f^n\|_{H_{l+\gamma/2+2}^{N+1}} \|\partial_v^\alpha h^n\|_{\epsilon, l+\gamma/2},$$

and so for any $\eta > 0$,

$$\mathfrak{I}_2(\alpha_1, \alpha_2) - \eta \|\partial_v^\alpha h^n\|_{\epsilon, l+\gamma/2}^2 \lesssim C(\eta, \|f_0\|_{H_{l+\gamma+2}^{N+2}}, \|f_0\|_{L_{q(N+1, l+\gamma/2+2)+\gamma}^1}) \|h^{n-1}\|_{H_{2l+3}^N}^2.$$

Taking a suitable η , we obtain

$$\begin{aligned} &\frac{d}{dt} \left(\frac{1}{2} \|\partial_v^\alpha h^n\|_{L_t^2}^2 \right) + \frac{C_1(f_0)}{4} \|\partial_v^\alpha h^n\|_{\epsilon, l+\gamma/2}^2 \\ &\lesssim C(\|f_0\|_{H_{2l+3+\gamma/2}^{N+1}}, \|f_0\|_{L_{q(N, 2l+3)+\gamma}^1}) \|h^n\|_{H_{l+\gamma/2+2}^N}^2 \\ &\quad + C(\|f_0\|_{H_{l+\gamma+2}^{N+2}}, \|f_0\|_{L_{q(N+1, l+\gamma/2+2)+\gamma}^1}) \|h^{n-1}\|_{H_{2l+3}^N}^2. \end{aligned}$$

Now taking sum over $|\alpha| \leq N$, we arrive at

$$\begin{aligned} &\frac{d}{dt} \|h^n\|_{H_l^N}^2 + \frac{C_1(f_0)}{2} \|h^n\|_{H_{l+\gamma/2}^{N+s}}^2 \\ &\lesssim C(\|f_0\|_{H_{2l+3+\gamma/2}^{N+1}}, \|f_0\|_{L_{q(N, 2l+3)+\gamma}^1}) \|h^n\|_{H_{l+\gamma/2+2}^N}^2 \\ &\quad + C(\|f_0\|_{H_{l+\gamma+2}^{N+2}}, \|f_0\|_{L_{q(N+1, l+\gamma/2+2)+\gamma}^1}) \|h^{n-1}\|_{H_{2l+3}^N}^2. \end{aligned}$$

By interpolation theory, one has

$$\|h^n\|_{H_{l+\gamma/2+2}^N}^2 \leq \eta \|h^n\|_{H_{l+\gamma/2}^{N+s}}^2 + c(\eta) \|h^n\|_{L_{w_1(N, l, s, \gamma)}^1}^2,$$

and

$$\|h^{n-1}\|_{H_{2l+3}^N}^2 \leq \lambda \|h^{n-1}\|_{H_{l+\gamma/2}^{N+s}}^2 + c(\lambda) \|h^{n-1}\|_{L_{w_2(N, l, s, \gamma)}^1}^2,$$

where $w_1(N, l, s, \gamma) = l + \gamma/2 + \frac{2(N+s+2)}{s}$ and $w_2(N, l, s, \gamma) = \frac{(N+s+2)(2l+3)-(N+2)(l+\gamma/2)}{s}$. It is easy to check $w_1 \leq w_2$. Choosing suitable η and λ , we have

$$\begin{aligned} & \frac{d}{dt} \|h^n\|_{H_l^N}^2 + \frac{C_1(f_0)}{4} \|h^n\|_{H_{l+\gamma/2}^{N+s}}^2 \\ & \leq \left(\frac{8}{9}\right) \frac{C_1(f_0)}{4} \|h^{n-1}\|_{H_{l+\gamma/2}^{N+s}}^2 \\ & \quad + C(\|f_0\|_{H_{2l+3+\gamma/2}^{N+1}}, \|f_0\|_{L_{q(N, 2l+3)+\gamma}^1}) \|h^n\|_{L_{w_1(N, l, s, \gamma)}^1}^2 \\ & \quad + C(\|f_0\|_{H_{l+\gamma+2}^{N+2}}, \|f_0\|_{L_{q(N+1, l+\gamma/2+2)+\gamma}^1}) \|h^{n-1}\|_{L_{w_2(N, l, s, \gamma)}^1}^2. \end{aligned}$$

Thanks to (3.48), on the time interval $[0, T^*(q(2, w_2 + \gamma + 4))]$, there holds

$$\|h^n(t)\|_{L_{w_2}^1} \leq \left(\frac{8}{9}\right)^n M(T^*) \exp\left(\frac{9K_2 T^*}{8} + K_1 T^*\right) \stackrel{\text{def}}{=} \left(\frac{8}{9}\right)^n M$$

where $T^* = T^*(q(w_2 + \gamma + 4, 2))$ and $M(T^*) = \frac{9}{8} A_2^s m(0) \int_0^{T^*} e^{-K_1 s} \|h^0(s)\|_{L_{w_2+\gamma}^1} ds + 22m(w_2)$. For ease of notation, let $K_3 = C(\eta, \|f_0\|_{H_{2l+3+\gamma/2}^{N+1}}, \|f_0\|_{L_{q(N, 2l+3)+\gamma}^1})$ and $K_4 = C(\lambda, \|f_0\|_{H_{l+\gamma+2}^{N+2}}, \|f_0\|_{L_{q(N+1, l+\gamma/2+2)+\gamma}^1})$. Then we have

$$\begin{aligned} & \frac{d}{dt} \|h^n\|_{H_l^N}^2 + \frac{C_1(f_0)}{4} \|h^n\|_{H_{l+\gamma/2}^{N+s}}^2 \\ & \leq \left(\frac{8}{9}\right) \frac{C_1(f_0)}{4} \|h^{n-1}\|_{H_{l+\gamma/2}^{N+s}}^2 + M(K_3 + \frac{9}{8} K_4) \left(\frac{8}{9}\right)^n. \end{aligned}$$

Integrating both sides with respect to time over $[0, t]$ for any $t \in [0, T^*]$, we have

$$\begin{aligned} & \|h^n(t)\|_{H_l^N}^2 + \frac{C_1(f_0)}{4} \int_0^t \|h^n(r)\|_{H_{l+\gamma/2}^{N+s}}^2 dr \\ & \leq \left(\frac{8}{9}\right) \frac{C_1(f_0)}{4} \int_0^t \|h^{n-1}(r)\|_{H_{l+\gamma/2}^{N+s}}^2 dr + MT^*(K_3 + \frac{9}{8} K_4) \left(\frac{8}{9}\right)^n \\ & \leq \left(\frac{8}{9}\right)^n \frac{C_1(f_0)}{4} \int_0^t \|h^0(r)\|_{H_{l+\gamma/2}^{N+s}}^2 dr + MT^*(K_3 + \frac{9}{8} K_4) \left(\frac{8}{9}\right)^n. \end{aligned}$$

Thus $\sum_n \|h^n(t)\|_{H_l^N}$ is finite and $\{f^n(t)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in H_l^N . So there is a function $f \in L^\infty([0, T^*]; H_l^N)$ such that

$$\lim_{n \rightarrow \infty} \sup_{0 \leq t \leq T^*} \|f^n(t) - f(t)\|_{H_l^N} = 0.$$

The condition on f_0 can be summarized by the definitions of K_3, K_4 and the previous step as

$$f_0 \in H_{w_H(N, l)}^{(N+2) \vee 3} \cap L_{w_L(N, l)}^1.$$

Under this condition, actually $\{f^n(t)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $H_l^N \cap L_{w(N, l)}^1$.

It is obvious that f is the solution to (1.7). Because f^n is non-negative, the limit function f is also non-negative.

Step 4: (Uniqueness)

Suppose $f, g \in L^\infty([0, T]; H_{l+\gamma+4}^2)$ are two non-negative solutions to (1.7). Set $F = f - g$ and $G = f + g$. Then F is a solution to the following equation,

$$(3.49) \quad \begin{cases} \partial_t F = M^\epsilon(G, F) + M^\epsilon(F, G) \\ F|_{t=0} = 0. \end{cases}$$

Note that the above equation is as the same as equation (3.42) if $h^n = h^{n-1}$. Thus following the same argument until inequality (3.47), we have

$$\frac{d}{dt} \|F\|_{L_l^1} + \frac{1}{8} A_2^\epsilon \|G\|_{L^1} \|F\|_{L_{l+\gamma}^1} \leq K \|F\|_{L_l^1}$$

where K is some constant depending on the uniform upper bound of $\|G\|_{H_{l+\gamma+4}^2}$. Note that the previous estimate holds true for $l \geq 4$. Therefore, our approximate equation (1.7) has at most one solution in the space $L^\infty([0, T]; H_l^N)$ if $N \geq 2$ and $l \geq 8 + \gamma$. \square

3.3. Improvement of the well-posedness result of approximate equation (1.7). In this subsection, by using the symmetric property of the collision operators, we will prove the propagation of L_i^1 and H_i^N norms of the solution f to (1.7) and then extend the lifespan T^* in Lemma 3.3 to be global. Thanks to Lemma 3.3, we may assume that solution f^ϵ to our approximate equation is non-negative and smooth. It means that in this subsection we only need to give the *a priori* estimates to the equation.

In order to prove the propagation of L_i^1 of the solution f^ϵ , we first give two propositions. The first proposition is related to the Boltzmann operator, while the second deals with the Landau operator.

Proposition 3.2. *Let $p \geq 3$ and $k_p = [\frac{p+1}{2}]$. Suppose*

$$(3.50) \quad \Theta(v, v_*) \stackrel{\text{def}}{=} \int_{SS^2} b(\cos \theta) |v - v_*|^\gamma (\langle v' \rangle^{2p} + \langle v'_* \rangle^{2p} - \langle v \rangle^{2p} - \langle v_* \rangle^{2p}) d\sigma,$$

then one has

$$\begin{aligned} \Theta(v, v_*) &\leq -\frac{1}{4} A_2 (\langle v \rangle^{2p+\gamma} + \langle v_* \rangle^{2p+\gamma}) + \frac{1}{2} A_2 (\langle v \rangle^{2p} \langle v_* \rangle^\gamma + \langle v \rangle^\gamma \langle v_* \rangle^{2p}) \\ &\quad + A_2 \sum_{k=1}^{k_p} \binom{p}{k} \{ \langle v \rangle^{2k+\gamma} \langle v_* \rangle^{2(p-k)} + \langle v \rangle^{2(p-k)+\gamma} \langle v_* \rangle^{2k} + \\ &\quad + \langle v \rangle^{2(p-k)} \langle v_* \rangle^{2k+\gamma} + \langle v \rangle^{2k} \langle v_* \rangle^{2(p-k)+\gamma} \} \\ &\quad + 2p(p-1) A_2 \sum_{k=0}^{k_p-1} \binom{p-2}{k} \{ \langle v \rangle^{2(k+1)+\gamma} \langle v_* \rangle^{2(p-k-1)} + \langle v \rangle^{2(p-k-1)+\gamma} \langle v_* \rangle^{2(k+1)} \\ &\quad + \langle v \rangle^{2(p-k-1)} \langle v_* \rangle^{2(k+1)+\gamma} + \langle v \rangle^{2(k+1)} \langle v_* \rangle^{2(p-k-1)+\gamma} \} \\ &\leq -\frac{1}{4} A_2 (\langle v \rangle^{2p+\gamma} + \langle v_* \rangle^{2p+\gamma}) + 2^{2p+1} A_2 \langle v \rangle^{2p} \langle v_* \rangle^{2p}. \end{aligned}$$

Proof. One may refer to Lemma 3.6 in [14] for the proof. \square

Remark 3.1. *Lemma 3.6 in [14] only deals with the case $p \geq 3$, however, the conclusion is also valid in the case $2 \leq p < 3$ but with a different and smaller coefficient coming out instead of the constant $\frac{1}{4}$ before the highest order $2p + \gamma$.*

Proposition 3.3. *Let $p > 2$ and f be a non-negative function, then*

$$(3.51) \quad \langle Q_L(f, f), \langle v \rangle^p \rangle \leq -\Lambda p \|f\|_{L^1} \|f\|_{L_{p+\gamma}^1} + \Lambda p(4p+2) \|f\|_{L_2^1} \|f\|_{L_p^1}.$$

Proof. One may refer to [6] for the proof. \square

Now we are ready to prove the propagation of moments and the smoothness.

Proof of Theorem 1.1: The proof will be divided into four steps.

Step 1: Propagation of the moments.

We consider the $2l$ moment. Assume $l \geq 3$, for the case $2 \leq l < 3$, the proof is similar thanks to remark 3.1. By the case By the definition of M^ϵ , we have

$$\begin{aligned} \frac{d}{dt} \|f^\epsilon\|_{L_{2l}^1} &= \langle Q^\epsilon(f^\epsilon, f^\epsilon), \langle v \rangle^{2l} \rangle + \epsilon^{2-2s} \langle Q_L(f^\epsilon, f^\epsilon), \langle v \rangle^{2l} \rangle \\ &\stackrel{\text{def}}{=} \mathfrak{J}_1 + \mathfrak{J}_2. \end{aligned}$$

The term \mathfrak{J}_1 can be written as:

$$\begin{aligned} \mathfrak{J}_1 &= \int_{\mathbb{R}^6 \times SS^2} b^\epsilon(\cos \theta) |v - v_*|^\gamma f_*^\epsilon f^\epsilon (\langle v' \rangle^{2l} - \langle v \rangle^{2l}) d\sigma dv_* dv \\ &= \frac{1}{2} \int_{\mathbb{R}^6 \times SS^2} b^\epsilon(\cos \theta) |v - v_*|^\gamma f_*^\epsilon f^\epsilon (\langle v' \rangle^{2l} + \langle v'_* \rangle^{2l} - \langle v \rangle^{2l} - \langle v_* \rangle^{2l}) d\sigma dv_* dv \end{aligned}$$

Let $A_2^\epsilon = \int_{S^2} b^\epsilon(\cos \theta) \sin^2 \theta d\sigma$, then by proposition 3.2, we have

$$\begin{aligned} \mathfrak{I}_1 &\leq -\frac{A_2^\epsilon}{4} \|f^\epsilon\|_{L_0^1} \|f^\epsilon\|_{L_{2l+\gamma}^1} + \frac{A_2^\epsilon}{2} \|f^\epsilon\|_{L_2^1} \|f^\epsilon\|_{L_{2l}^1} \\ &\quad + A_2^\epsilon \sum_{k=1}^{k_l} \binom{l}{k} \{ \|f^\epsilon\|_{L_{2k+2}^1} \|f^\epsilon\|_{L_{2(l-k)}^1} + \|f^\epsilon\|_{L_{2k}^1} \|f^\epsilon\|_{L_{2(l-k)+2}^1} \} \\ &\quad + 2l(l-1) A_2^\epsilon \sum_{k=0}^{k_l-1} \binom{l-2}{k} \{ \|f^\epsilon\|_{L_{2(k+1)+2}^1} \|f^\epsilon\|_{L_{2(l-k-1)}^1} + \|f^\epsilon\|_{L_{2(l-k)}^1} \|f^\epsilon\|_{L_{2(k+1)}^1} \}, \end{aligned}$$

where we have used the assumption $\gamma \leq 2$. By interpolation, for any $2 \leq p, q \leq 2l$ with $p+q = 2l+2$, we have

$$\|f^\epsilon\|_{L_p^1} \|f^\epsilon\|_{L_q^1} \leq \|f^\epsilon\|_{L_2^1} \|f^\epsilon\|_{L_{2l}^1}.$$

Using the fact $2 \sum_{k=1}^{k_l} \binom{l}{k} \leq 2^l$, we can conclude:

$$(3.52) \quad \mathfrak{I}_1 \leq -\frac{A_2^\epsilon}{4} \|f^\epsilon\|_{L_0^1} \|f^\epsilon\|_{L_{2l+\gamma}^1} + 2^{2l+1} A_2^\epsilon \|f^\epsilon\|_{L_2^1} \|f^\epsilon\|_{L_{2l}^1}.$$

For the term \mathfrak{I}_2 , we apply proposition 3.3 with $p = 2l$ and obtain

$$\mathfrak{I}_2 \leq -2l\Lambda\epsilon^{2-2s} \|f\|_{L^1} \|f\|_{L_{2l+\gamma}^1} + 2l(8l+2)\Lambda\epsilon^{2-2s} \|f\|_{L_2^1} \|f\|_{L_{2l}^1}.$$

Let $0 < \epsilon_* < \frac{\sqrt{2}}{2}$ be the point such that $A_2^* = \frac{A_2}{2}$, then for any $0 < \epsilon \leq \epsilon_*$, we have

$$\frac{d}{dt} \|f^\epsilon\|_{L_{2l}^1} \leq -\frac{A_2}{8} \|f^\epsilon\|_{L^1} \|f^\epsilon\|_{L_{2l+\gamma}^1} + (2^{2l+1} A_2 + 2l(8l+2)\Lambda) \|f^\epsilon\|_{L_2^1} \|f^\epsilon\|_{L_{2l}^1}.$$

For any $\eta > 0$, there exists a constant $K_1(\eta, l)$ such that

$$\langle v \rangle^{2l} \leq \eta \langle v \rangle^{2l+\gamma} + K_1(\eta, l).$$

Thus we have $\|f^\epsilon\|_{L_{2l}^1} \leq \eta \|f^\epsilon\|_{L_{2l+\gamma}^1} + K_1(\eta, l) \|f^\epsilon\|_{L^1}$. With the preservation of mass and energy, by denoting $K_2(l) = 2^{2l+1} A_2 + 2l(8l+2)\Lambda$ and taking $\eta(f_0) = \frac{A \|f_0\|_{L^1}}{16K_2(l) \|f_0\|_{L_2^1}}$, we have

$$\frac{d}{dt} \|f^\epsilon\|_{L_{2l}^1} \leq -\frac{A_2}{16} \|f_0\|_{L^1} \|f^\epsilon\|_{L_{2l+\gamma}^1} + K_2(l) K_1(\eta(f_0), l) \|f_0\|_{L_2^1} \|f_0\|_{L^1}.$$

Let $a = K_2(l) K_1(\eta(f_0), l) \|f_0\|_{L_2^1} \|f_0\|_{L^1}$ and $b = -\frac{A_2}{16} \|f_0\|_{L^1}$, by Gronwall's inequality (1.11), we have the following:

$$\|f^\epsilon(t)\|_{L_{2l}^1} \leq \|f_0\|_{L_{2l}^1} + \frac{a}{|b|} \stackrel{\text{def}}{=} \|f_0\|_{L_{2l}^1} + K(f_0, l).$$

The constant $K(f_0, l)$ depends only on l , $\|f_0\|_{L^1}$ and $\|f_0\|_{L_2^1}$.

Step 2: Propagation of L_l^2 norm.

By the definition of M^ϵ , we have

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|f^\epsilon\|_{L_l^2}^2 \right) &= \langle M^\epsilon(f^\epsilon, f^\epsilon \langle v \rangle^l), f^\epsilon \langle v \rangle^l \rangle + \{ \langle M^\epsilon(f^\epsilon, f^\epsilon) \langle v \rangle^l - M^\epsilon(f^\epsilon, f^\epsilon \langle v \rangle^l), f^\epsilon \langle v \rangle^l \} \\ &\stackrel{\text{def}}{=} \mathfrak{I}_1 + \mathfrak{I}_2. \end{aligned}$$

Applying coercivity estimates of (2.5) with $g = f^\epsilon, f = f^\epsilon \langle v \rangle^l$, we have

$$(3.53) \quad \mathfrak{I}_1 \leq -C_1(f_0) \|f^\epsilon\|_{\epsilon, l+\gamma/2}^2 + C_2(f_0) \|f^\epsilon\|_{L_{l+\gamma/2}^2}^2.$$

Applying commutator estimates (2.9) with $g = f^\epsilon, h = f^\epsilon, f = f^\epsilon \langle v \rangle^l, N_2 = l + \gamma/2, N_3 = \gamma/2$ and $N_1 = 2l + 5$, we have

$$\mathfrak{I}_2 \lesssim \|f^\epsilon\|_{L_{2l+5}^1} (\|f^\epsilon\|_{H_{l+\gamma/2}^s} + \epsilon^{2-2s} \|f^\epsilon\|_{H_{l+\gamma/2}^1}) \|f^\epsilon\|_{L_{l+\gamma/2}^2}.$$

Thanks to the facts $\|\cdot\|_{H_{l+\gamma/2}^s}^2 \leq \|\cdot\|_{\epsilon, l+\gamma/2}^2$ and $\epsilon^{2-2s} \|\cdot\|_{H_{l+\gamma/2}^1}^2 \leq \|\cdot\|_{\epsilon, l+\gamma/2}^2$, we have

$$(3.54) \quad \mathfrak{I}_2 - \frac{C_1(f_0)}{2} \|f^\epsilon\|_{\epsilon, l+\gamma/2}^2 \lesssim \frac{1}{C_1(f_0)} \|f^\epsilon\|_{L_{2l+5}^1}^2 \|f^\epsilon\|_{L_{l+\gamma/2}^2}^2.$$

Now patching together (3.53), and (3.54), we get

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \|f^\epsilon\|_{L_t^2}^2 \right) + \frac{C_1(f_0)}{2} \|f^\epsilon\|_{\epsilon, l+\gamma/2}^2 \\ & \lesssim (C_2(f_0) + \frac{1}{C_1(f_0)} \|f^\epsilon\|_{L_{2l+5}^1}^2) \|f^\epsilon\|_{L_{l+\gamma/2}^2}^2 \\ & \lesssim C_3(\|f_0\|_{L_{2l+5}^1}, \|f_0\|_{L \log L}) \|f^\epsilon\|_{L_{l+\gamma/2}^2}^2, \end{aligned}$$

where the existence of $C_3(f_0, l) = C_3(\|f_0\|_{L_{2l+5}^1}, \|f_0\|_{L \log L}, l)$ is ensured by the previous step.

By applying (3.16) with $\lambda = \frac{C_1(f_0)}{4C_3(f_0, l)}$, we have

$$\frac{d}{dt} \left(\frac{1}{2} \|f^\epsilon\|_{L_t^2}^2 \right) + \frac{C_1(f_0)}{4} \|f^\epsilon\|_{\epsilon, l+\gamma/2}^2 \lesssim C_3(f_0, l) \left(\frac{C_1(f_0)}{4C_3(f_0, l)} \right)^{-\frac{3}{2s}} \|f^\epsilon\|_{L_{l+\gamma/2}^1}^2.$$

Thanks to Gronwall's inequality, there exists a constant $C(\|f_0\|_{L_{2l+5}^1}, \|f_0\|_{L_t^2})$ such that for any $t \geq 0$,

$$(3.55) \quad \|f^\epsilon(t)\|_{L_t^2}^2 + \int_t^{t+1} \|f^\epsilon(r)\|_{\epsilon, l+\gamma/2}^2 dr \leq C(\|f_0\|_{L_{2l+5}^1}, \|f_0\|_{L_t^2}).$$

Inequality (1.17) is obtained in the case of $N = 0$.

Step 3: Propagation of H_l^s norm.

We first introduce some notations for the fractional derivative. We set

$$\Delta_s f = \frac{(\tau_h f)(v) - f(v)}{|h|^{\frac{3}{2}+s}},$$

with $(\tau_h f)(v) = f(v+h)$ and $0 < s < 1$. Then there holds

$$\begin{aligned} \Delta_s(fg) &= \Delta_s f g + \tau_h f \Delta_s g \\ &= f \Delta_s g + \Delta_s f \tau_h g. \end{aligned}$$

Due to the definition of the fractional Sobolev space, we observe that:

$$(3.56) \quad \|g\|_{H^s}^2 \sim \int_{|h| \leq \frac{1}{2}} \|\Delta_s g\|_{L^2}^2 dh + \|g\|_{L^2}^2.$$

Moreover, we also have, for $|h| \leq \frac{1}{2}$ and $m \in \mathbb{R}$,

$$(3.57) \quad \|g \langle v \rangle^k \Delta_s \langle v \rangle^l\|_{H^m} \lesssim |h|^{-(\frac{1}{2}+s)} \|g \langle v \rangle^{l+k}\|_{H^m},$$

$$(3.58) \quad \|\tau_h g\|_{H_l^m} \sim \|g\|_{H_l^m},$$

and

$$\begin{aligned} & \|g\|_{H_l^m}^2 + \int_{|h| \leq \frac{1}{2}} \|\langle v \rangle^l \Delta_s g\|_{H^m}^2 dh \\ (3.59) \quad & \sim \|g\|_{H_l^m}^2 + \int_{|h| \leq \frac{1}{2}} \|\Delta_s(g \langle v \rangle^l)\|_{H^m}^2 dh \sim \|g\|_{H_l^{m+s}}^2. \end{aligned}$$

One may check the proof of (3.57), (3.58) and (3.59) in the appendix of [10].

Let $g^\epsilon = f^\epsilon \langle v \rangle^l$. It is easy to check that $\Delta_s g^\epsilon$ solves the following equation:

$$\begin{aligned} & \partial_t(\Delta_s g^\epsilon) - M^\epsilon(f^\epsilon, \Delta_s g^\epsilon) \\ &= M^\epsilon(\Delta_s f^\epsilon, \tau_h g^\epsilon) + [M^\epsilon(\Delta_s f^\epsilon, f^\epsilon \langle v \rangle^l) - M^\epsilon(\Delta_s f^\epsilon, f^\epsilon \langle v \rangle^l)] \\ & \quad + [M^\epsilon(\tau_h f^\epsilon, \Delta_s f^\epsilon \langle v \rangle^l) - M^\epsilon(\tau_h f^\epsilon, \Delta_s f^\epsilon \langle v \rangle^l)] \\ & \quad + [M^\epsilon(\tau_h f^\epsilon, \tau_h f^\epsilon) \Delta_s \langle v \rangle^l - M^\epsilon(\tau_h f^\epsilon, \tau_h f^\epsilon \Delta_s \langle v \rangle^l)] \\ & \stackrel{\text{def}}{=} \sum_{i=1}^4 F_i. \end{aligned}$$

By the upper bound estimate (2.1), noting $\gamma \leq 2$, we have

$$\langle F_1, \Delta_s g^\epsilon \rangle \lesssim \|\Delta_s f^\epsilon\|_{L_4^1} (\|g^\epsilon\|_{H_3^s} \|\Delta_s g^\epsilon\|_{H_{\gamma/2}^s} + \epsilon^{2-2s} \|g^\epsilon\|_{H_3^1} \|\Delta_s g^\epsilon\|_{H_{\gamma/2}^1}),$$

which implies, for any $\eta_1 > 0$,

$$(3.60) \quad \langle F_1, \Delta_s g^\epsilon \rangle - \eta_1 \|\Delta_s g^\epsilon\|_{\epsilon, \gamma/2}^2 \lesssim \frac{1}{2\eta_1} (\|\Delta_s f^\epsilon\|_{L_4^1}^2 \|g^\epsilon\|_{H_3^s}^2 + \epsilon^{2-2s} \|\Delta_s f^\epsilon\|_{L_4^1}^2 \|g^\epsilon\|_{H_3^1}^2).$$

By the commutator estimates (2.7) and (2.8), we have

$$\begin{aligned} \langle F_2, \Delta_s g^\epsilon \rangle &\lesssim \|\Delta_s f^\epsilon\|_{L_{2l+1}^1} \|f^\epsilon\|_{H_{l+\gamma/2}^s} \|\Delta_s g^\epsilon\|_{L_{\gamma/2}^2} \\ &\quad + \epsilon^{2-2s} \|\Delta_s f^\epsilon\|_{L_{\gamma+3}^1} \|f^\epsilon\|_{H_{l+\gamma/2}^1} \|\Delta_s g^\epsilon\|_{L_{\gamma/2}^2}, \end{aligned}$$

which implies, for any $\eta_1 > 0$,

$$(3.61) \quad \begin{aligned} &\langle F_2, \Delta_s g^\epsilon \rangle - \eta_1 \|\Delta_s g^\epsilon\|_{\epsilon, \gamma/2}^2 \\ &\lesssim \frac{1}{2\eta_1} (\|\Delta_s f^\epsilon\|_{L_{2l+1}^1}^2 \|f^\epsilon\|_{H_{l+1}^s}^2 + \epsilon^{2-2s} \|\Delta_s f^\epsilon\|_{L_5^1}^2 \|f^\epsilon\|_{H_{l+1}^1}^2). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \langle F_3, \Delta_s g^\epsilon \rangle &\lesssim \|f^\epsilon\|_{L_{2l+1}^1} \|\Delta_s f^\epsilon\|_{H_{l+\gamma/2}^s} \|\Delta_s g^\epsilon\|_{L_{\gamma/2}^2} \\ &\quad + \epsilon^{2-2s} \|f^\epsilon\|_{L_{\gamma+3}^1} \|\Delta_s f^\epsilon\|_{H_{l+\gamma/2}^1} \|\Delta_s g^\epsilon\|_{L_{\gamma/2}^2}, \end{aligned}$$

which implies, for any $\eta_2 > 0$,

$$(3.62) \quad \begin{aligned} &\langle F_3, \Delta_s g^\epsilon \rangle - \eta_2 \|\Delta_s f^\epsilon\|_{\epsilon, l+\gamma/2}^2 \\ &\lesssim \frac{1}{2\eta_2} (\|f^\epsilon\|_{L_{2l+1}^1}^2 \|\Delta_s g^\epsilon\|_{L_1^2}^2 + \epsilon^{2-2s} \|f^\epsilon\|_{L_5^1}^2 \|\Delta_s g^\epsilon\|_{L_1^2}^2). \end{aligned}$$

Also by the upper bound estimate (2.1), we have

$$\begin{aligned} \langle F_4, \Delta_s g^\epsilon \rangle &\lesssim \|f^\epsilon\|_{L_{l+5}^1} \|f^\epsilon\|_{H_{l+3}^s} \|(\Delta_s g^\epsilon)(\Delta_s \langle v \rangle^l)\|_{H_{-l+\gamma/2}^s} \\ &\quad + \epsilon^{2-2s} \|f^\epsilon\|_{L_{l+5}^1} \|f^\epsilon\|_{H_{l+3}^1} \|(\Delta_s g^\epsilon)(\Delta_s \langle v \rangle^l)\|_{H_{-l+\gamma/2}^1} \\ &\quad + \|f^\epsilon\|_{L_4^1} \|(\tau_h f^\epsilon)(\Delta_s \langle v \rangle^l)\|_{H_3^s} \|\Delta_s g^\epsilon\|_{H_{\gamma/2}^s} \\ &\quad + \epsilon^{2-2s} \|f^\epsilon\|_{L_4^1} \|(\tau_h f^\epsilon)(\Delta_s \langle v \rangle^l)\|_{H_3^1} \|\Delta_s g^\epsilon\|_{H_{\gamma/2}^1}, \end{aligned}$$

which implies, for any $\eta_1 > 0$,

$$(3.63) \quad \begin{aligned} &\langle F_4, \Delta_s g^\epsilon \rangle - \eta_1 \|\Delta_s g^\epsilon\|_{\epsilon, \gamma/2}^2 \\ &\lesssim \frac{1}{\eta_1} |h|^{-(1+2s)} (\|f^\epsilon\|_{L_{l+5}^1}^2 \|f^\epsilon\|_{H_{l+3}^s}^2 + \epsilon^{2-2s} \|f^\epsilon\|_{L_{l+5}^1}^2 \|f^\epsilon\|_{H_{l+3}^1}^2). \end{aligned}$$

By the coercivity estimate (2.5), we have

$$(3.64) \quad \langle M^\epsilon(f^\epsilon, \Delta_s g^\epsilon), \Delta_s g^\epsilon \rangle \leq -C_1(f_0) \|\Delta_s g^\epsilon\|_{\epsilon, \gamma/2}^2 + C_2(f_0) \|\Delta_s g^\epsilon\|_{L_{\gamma/2}^2}^2.$$

Thanks to

$$\frac{d}{dt} \left(\frac{1}{2} \|\Delta_s g^\epsilon\|_{L^2}^2 \right) = \langle \partial_t \Delta_s g^\epsilon, \Delta_s g^\epsilon \rangle = \langle M^\epsilon(f^\epsilon, \Delta_s g^\epsilon), \Delta_s g^\epsilon \rangle + \sum_{i=1}^4 \langle F_i, \Delta_s g^\epsilon \rangle,$$

patching together all the above estimates, taking $\eta_1 = \frac{C_1(f_0)}{6}$ in (3.60),(3.61),(3.63), we arrive at, for $|h| \leq \frac{1}{2}$,

$$\begin{aligned}
& \frac{d}{dt} \left(\frac{1}{2} \|\Delta_s g^\epsilon\|_{L^2}^2 \right) + \frac{C_1(f_0)}{2} \|\Delta_s g^\epsilon\|_{\epsilon, \gamma/2}^2 - \eta_2 \|\Delta_s f^\epsilon\|_{\epsilon, l+\gamma/2}^2 \\
& \lesssim +C_2(f_0) \|\Delta_s g^\epsilon\|_{L^2_{\gamma/2}}^2 + \frac{1}{2\eta_1} (\|\Delta_s f^\epsilon\|_{L^1_4}^2 \|g^\epsilon\|_{H^s_3}^2 + \epsilon^{2-2s} \|\Delta_s f^\epsilon\|_{L^1_4}^2 \|g^\epsilon\|_{H^1_3}^2) \\
& + \frac{1}{2\eta_1} (\|\Delta_s f^\epsilon\|_{L^1_{2l+1}}^2 \|f^\epsilon\|_{H^s_{l+1}}^2 + \epsilon^{2-2s} \|\Delta_s f^\epsilon\|_{L^1_5}^2 \|f^\epsilon\|_{H^1_{l+1}}^2) \\
& + \frac{1}{2\eta_2} (\|f^\epsilon\|_{L^1_{2l+1}}^2 \|\Delta_s g^\epsilon\|_{L^2_1}^2 + \epsilon^{2-2s} \|f^\epsilon\|_{L^1_5}^2 \|\Delta_s g^\epsilon\|_{L^2_1}^2) \\
& + \frac{1}{\eta_1} |h|^{-(1+2s)} (\|f^\epsilon\|_{L^1_{l+5}}^2 \|f^\epsilon\|_{H^s_{l+3}}^2 + \epsilon^{2-2s} \|f^\epsilon\|_{L^1_{l+5}}^2 \|f^\epsilon\|_{H^1_{l+3}}^2) \\
& \lesssim +C_2(f_0) \|\Delta_s g^\epsilon\|_{L^2_{\gamma/2}}^2 + \frac{1}{\eta_1} \|\Delta_s f^\epsilon\|_{L^2_6}^2 \|g^\epsilon\|_{\epsilon, 3}^2 \\
& + \frac{1}{\eta_1} (\|\Delta_s f^\epsilon\|_{L^2_{2l+3}}^2 \|f^\epsilon\|_{H^s_{l+1}}^2 + \epsilon^{2-2s} \|\Delta_s f^\epsilon\|_{L^2_7}^2 \|f^\epsilon\|_{H^1_{l+1}}^2) \\
& + \frac{1}{\eta_2} \|f^\epsilon\|_{L^1_{2l+5}}^2 \|\Delta_s g^\epsilon\|_{L^2_1}^2 + \frac{1}{\eta_1} |h|^{-(1+2s)} \|f^\epsilon\|_{L^1_{l+5}}^2 \|f^\epsilon\|_{\epsilon, l+3}^2.
\end{aligned}$$

where we have used the fact $\|\langle \cdot \rangle^{-2}\|_{L^2} \leq \sqrt{2\pi}$. Integrating both sides from 0 to t with respect to time, we obtain

$$\begin{aligned}
(3.65) \quad & \|\Delta_s g^\epsilon(t)\|_{L^2}^2 + C_1(f_0) \int_0^t \|\Delta_s g^\epsilon(r)\|_{\epsilon, \gamma/2}^2 dr - 2\eta_2 \int_0^t \|\Delta_s f^\epsilon(r)\|_{\epsilon, l+\gamma/2}^2 dr \\
& \lesssim \|\Delta_s g^\epsilon(0)\|_{L^2}^2 + C_2(f_0) \int_0^t \|\Delta_s g^\epsilon(r)\|_{L^2_{\gamma/2}}^2 dr + \frac{1}{\eta_1} \int_0^t \|\Delta_s f^\epsilon(r)\|_{L^2_6}^2 \|g^\epsilon(r)\|_{\epsilon, 3}^2 dr \\
& + \frac{1}{\eta_1} \int_0^t (\|\Delta_s f^\epsilon(r)\|_{L^2_{2l+3}}^2 \|f^\epsilon(r)\|_{H^s_{l+1}}^2 + \epsilon^{2-2s} \|\Delta_s f^\epsilon(r)\|_{L^2_7}^2 \|f^\epsilon(r)\|_{H^1_{l+1}}^2) dr \\
& + \frac{1}{\eta_2} \int_0^t \|f^\epsilon(r)\|_{L^1_{2l+5}}^2 \|\Delta_s g^\epsilon(r)\|_{L^2_1}^2 dr + \frac{1}{\eta_1} |h|^{-(1+2s)} \int_0^t \|f^\epsilon(r)\|_{L^1_{l+5}}^2 \|f^\epsilon(r)\|_{\epsilon, l+3}^2 dr.
\end{aligned}$$

Integrating both sides on the Ball $B(0, \frac{1}{2})$ with respect to the variable h , noting that $\int_{|h| \leq \frac{1}{2}} |h|^{-(1+2s)} dh$ is finite, thanks to the facts (3.56) and (3.59), taking a small enough η_2 , we derive that

$$\begin{aligned}
(3.66) \quad & \|g^\epsilon(t)\|_{H^s}^2 + \frac{C_1(f_0)}{2} \int_0^t \int_{|h| \leq \frac{1}{2}} \|\Delta_s g^\epsilon(r)\|_{\epsilon, \gamma/2}^2 dh dr \\
& \lesssim \|g^\epsilon(0)\|_{H^s}^2 + \|g^\epsilon(t)\|_{L^2}^2 + C_2(f_0) \int_0^t \|g^\epsilon(r)\|_{H^s_1}^2 dr \\
& + \frac{1}{\eta_1} \int_0^t \|f^\epsilon(r)\|_{H^s_6}^2 \|g^\epsilon(r)\|_{\epsilon, 3}^2 dr \\
& + \frac{1}{\eta_1} \int_0^t (\|f^\epsilon(r)\|_{H^s_{2l+3}}^2 \|f^\epsilon(r)\|_{H^s_{l+1}}^2 + \epsilon^{2-2s} \|f^\epsilon(r)\|_{H^s_7}^2 \|f^\epsilon(r)\|_{H^1_{l+1}}^2) dr \\
& + \frac{1}{\eta_2} \int_0^t \|f^\epsilon(r)\|_{L^1_{2l+5}}^2 \|g^\epsilon(r)\|_{H^s_1}^2 dr + \frac{1}{\eta_1} \int_0^t \|f^\epsilon(r)\|_{L^1_{l+5}}^2 \|f^\epsilon(r)\|_{\epsilon, l+3}^2 dr.
\end{aligned}$$

Using the fact $\|f^\epsilon\|_{H^s_{2l+3}}^2 \|f^\epsilon\|_{H^s_{l+1}}^2 \leq \|f^\epsilon\|_{H^s_l}^2 \|f^\epsilon\|_{H^s_{2l+4}}^2$, substituting into the uniform bound of $\|f^\epsilon\|_{L^1_{2l+5}}$ and $\|f^\epsilon\|_{L^2_1}$, we have

$$\begin{aligned}
(3.67) \quad & \|f^\epsilon(t)\|_{H^s_l}^2 + \frac{C_1(f_0)}{2} \int_0^t \int_{|h| \leq \frac{1}{2}} \|\Delta_s g^\epsilon(r)\|_{\epsilon, \gamma/2}^2 dh dr \\
& \lesssim \|f_0\|_{H^s_l}^2 + C(\|f_0\|_{L^2_l}, \|f_0\|_{L^1_{2l+5}}) + C(\|f_0\|_{L^1_{2l+5}}, \|f_0\|_{L \log L}) \int_0^t \|f^\epsilon(r)\|_{\epsilon, l+3}^2 dr \\
& + C(\|f_0\|_{L^1_1}, \|f_0\|_{L \log L}) \int_0^t \|f^\epsilon(r)\|_{H^s_l}^2 \|f^\epsilon(r)\|_{\epsilon, 2l+4}^2 dr.
\end{aligned}$$

Actually, inequality (3.67) holds true on any bounded interval. Therefore, for any $t_1 < t_2$ with $t_2 - t_1 \leq 2$, we have

$$\begin{aligned}
(3.68) \quad & \|f^\epsilon(t_2)\|_{H_l^s}^2 + \frac{C_1(f_0)}{2} \int_{t_1}^{t_2} \int_{|h| \leq \frac{1}{2}} \|\Delta_s g^\epsilon(r)\|_{\epsilon, \gamma/2}^2 dh dr \\
& \lesssim \|f^\epsilon(t_1)\|_{H_l^s}^2 + C(\|f_0\|_{L_l^2}, \|f_0\|_{L_{2l+5}^1}) + C(\|f_0\|_{L_{2l+5}^1}, \|f_0\|_{L \log L}) \int_{t_1}^{t_2} \|f^\epsilon(r)\|_{\epsilon, l+3}^2 dr \\
& \quad + C(\|f_0\|_{L_l^1}, \|f_0\|_{L \log L}) \int_{t_1}^{t_2} \|f^\epsilon(r)\|_{H_l^s}^2 \|f^\epsilon(r)\|_{\epsilon, 2l+4}^2 dr.
\end{aligned}$$

By Gronwall's inequality (1.12) and uniform upper bound (3.55) for integral of $\|f^\epsilon\|_{\epsilon, l}^2$ on any bounded interval, we arrive at

$$(3.69) \quad \|f^\epsilon(t_2)\|_{H_l^s}^2 \lesssim C(\|f_0\|_{L_{4l+13}^1}, \|f_0\|_{L_{2l+4}^2}) \{\|f^\epsilon(t_1)\|_{H_l^s}^2 + C(\|f_0\|_{L_{2l+11}^1}, \|f_0\|_{L_{l+3}^2})\}.$$

Also from (3.55), we conclude that, in any unit interval $[t, t+1]$, there exists at least one point t_* such that

$$(3.70) \quad \|f^\epsilon(t_*)\|_{H_l^s}^2 \lesssim C(\|f_0\|_{L_{2l+5}^1}, \|f_0\|_{L_l^2}).$$

Combining (3.69) and (3.70), we have

$$(3.71) \quad \|f^\epsilon(t)\|_{H_l^s}^2 \lesssim C(\|f_0\|_{H_l^s}, \|f_0\|_{L_{4l+13}^1}, \|f_0\|_{L_{2l+4}^2}).$$

Together with (3.68), we finally arrive at

$$\begin{aligned}
(3.72) \quad & \|f^\epsilon(t)\|_{H_l^s}^2 + \frac{C_1(f_0)}{2} \int_t^{t+1} \int_{|h| \leq \frac{1}{2}} \|\Delta_s g^\epsilon(r)\|_{\epsilon, \gamma/2}^2 dh dr \\
& \lesssim C(\|f_0\|_{H_l^s}, \|f_0\|_{L_{4l+13}^1}, \|f_0\|_{L_{2l+4}^2}).
\end{aligned}$$

By interpolation theory, there holds

$$\|f_0\|_{L_{2l+4}^2} \lesssim \|f_0\|_{H_l^s} + \|f_0\|_{H_{\phi(s,l)}^{-2}} \lesssim \|f_0\|_{H_l^s} + \|f_0\|_{L_{\phi(s,l)}^1},$$

where $\phi(s, l) = \frac{(2l+4)(2+s)-2l}{s} \geq 4l+13$. Therefore we have

$$(3.73) \quad \|f^\epsilon(t)\|_{H_l^s}^2 \lesssim C(\|f_0\|_{H_l^s}, \|f_0\|_{L_{\phi(s,l)}^1}).$$

Step 4: Propagation of H_l^N norm when $N \geq 1$.

We prove the propagation by induction on N . Let $m \geq 1$ be an integer. Suppose inequality (1.17) holds true for all $N \leq m-1$, we now prove that it is also valid for $N = m$.

Set $g^\epsilon = \partial_v^\alpha f^\epsilon \langle v \rangle^l$ with $|\alpha| \leq m$, then g^ϵ solves

$$\begin{aligned}
(3.74) \quad \partial_t g^\epsilon &= M^\epsilon(f^\epsilon, g^\epsilon) + [M^\epsilon(f^\epsilon, \partial_v^\alpha f^\epsilon) \langle v \rangle^l - M^\epsilon(f^\epsilon, \partial_v^\alpha f^\epsilon \langle v \rangle^l)] \\
&+ \sum_{|\alpha_1| \geq 1, \alpha_1 + \alpha_2 = \alpha} \binom{\alpha}{\alpha_1} M^\epsilon(\partial_v^{\alpha_1} f^\epsilon, \partial_v^{\alpha_2} f^\epsilon) \langle v \rangle^l.
\end{aligned}$$

By the coercivity estimate (2.5), we have

$$(3.75) \quad \langle M^\epsilon(f^\epsilon, g^\epsilon), g^\epsilon \rangle \leq -C_1(f_0) \|g^\epsilon\|_{\epsilon, \gamma/2}^2 + C_2(f_0) \|g^\epsilon\|_{L_{\gamma/2}^2}^2.$$

By the commutator estimate (2.9), we have

$$|\langle M^\epsilon(f^\epsilon, \partial_v^\alpha f^\epsilon) \langle v \rangle^l - M^\epsilon(f^\epsilon, \partial_v^\alpha f^\epsilon \langle v \rangle^l), g^\epsilon \rangle| \lesssim \|f^\epsilon\|_{L_{2l+5}^1} \|g^\epsilon\|_{\epsilon, \gamma/2} \|g^\epsilon\|_{L_{\gamma/2}^2},$$

which implies, for any $\eta_1 > 0$,

$$\begin{aligned}
(3.76) \quad & |\langle M^\epsilon(f^\epsilon, \partial_v^\alpha f^\epsilon) \langle v \rangle^l - M^\epsilon(f^\epsilon, \partial_v^\alpha f^\epsilon \langle v \rangle^l), g^\epsilon \rangle| - \eta_1 \|g^\epsilon\|_{\epsilon, \gamma/2}^2 \\
& \lesssim \frac{1}{\eta_1} \|f^\epsilon\|_{L_{2l+5}^1}^2 \|g^\epsilon\|_{L_{\gamma/2}^2}^2.
\end{aligned}$$

For the remaining terms in the right hand of (3.74) with $|\alpha_1| \geq 1$, we split each of them into two terms:

$$\begin{aligned} M^\epsilon(\partial_v^{\alpha_1} f^\epsilon, \partial_v^{\alpha_2} f^\epsilon) \langle v \rangle^l &= \{M^\epsilon(\partial_v^{\alpha_1} f^\epsilon, \partial_v^{\alpha_2} f^\epsilon) \langle v \rangle^l - M^\epsilon(\partial_v^{\alpha_1} f^\epsilon, \partial_v^{\alpha_2} f^\epsilon \langle v \rangle^l)\} \\ &\quad + M^\epsilon(\partial_v^{\alpha_1} f^\epsilon, \partial_v^{\alpha_2} f^\epsilon \langle v \rangle^l) \\ &\stackrel{\text{def}}{=} \mathfrak{I}_1 + \mathfrak{I}_2. \end{aligned}$$

By the commutator estimate (2.9), for the case $|\alpha_1| = |\alpha| \leq m$, we have

$$|\langle \mathfrak{I}_1, g^\epsilon \rangle| \lesssim \|\partial_v^{\alpha_1} f^\epsilon\|_{L_{2l+5}^1} \|f^\epsilon\|_{\epsilon, l+\gamma/2} \|g^\epsilon\|_{L_{\gamma/2}^2} \lesssim \|f^\epsilon\|_{H_{2l+7}^m} \|f^\epsilon\|_{\epsilon, l+\gamma/2} \|g^\epsilon\|_{L_{\gamma/2}^2},$$

which implies, for any $\eta_2 > 0$,

$$(3.77) \quad |\langle \mathfrak{I}_1, g^\epsilon \rangle| - \eta_2 \|g^\epsilon\|_{\epsilon, \gamma/2}^2 \lesssim \frac{1}{\eta_2} \|f^\epsilon\|_{H_{2l+7}^m}^2 \|f^\epsilon\|_{\epsilon, l+\gamma/2}^2$$

For the case $1 \leq |\alpha_1| \leq |\alpha| - 1 \leq m - 1$, we have

$$\begin{aligned} |\langle \mathfrak{I}_1, g^\epsilon \rangle| &\lesssim \|\partial_v^{\alpha_1} f^\epsilon\|_{L_{2l+5}^1} \|\partial_v^{\alpha_2} f^\epsilon\|_{\epsilon, l+\gamma/2} \|g^\epsilon\|_{L_{\gamma/2}^2} \\ &\lesssim (\|f^\epsilon\|_{H_{2l+7}^1} \|f^\epsilon\|_{H_{l+\gamma/2}^m} \mathbf{1}_{m \geq 2} + \|f^\epsilon\|_{H_{2l+7}^{m-1}} \|f^\epsilon\|_{H_{l+\gamma/2}^{m-1}}) \|g^\epsilon\|_{L_{\gamma/2}^2}, \end{aligned}$$

which implies, for any $\eta_2 > 0$,

$$\begin{aligned} &|\langle \mathfrak{I}_1, g^\epsilon \rangle| - \eta_2 \|g^\epsilon\|_{\epsilon, \gamma/2}^2 \\ &\lesssim \frac{1}{\eta_2} (\|f^\epsilon\|_{H_{2l+7}^1}^2 \|f^\epsilon\|_{H_{l+\gamma/2}^m}^2 \mathbf{1}_{m \geq 2} + \|f^\epsilon\|_{H_{2l+7}^{m-1}}^2 \|f^\epsilon\|_{H_{l+\gamma/2}^{m-1}}^2). \end{aligned}$$

By the upper bound estimate (2.1), for the case $|\alpha_1| = |\alpha| \leq m$, we have,

$$|\langle \mathfrak{I}_2, g^\epsilon \rangle| \lesssim \|\partial_v^{\alpha_1} f^\epsilon\|_{L_4^1} (\|f^\epsilon\|_{H_{l+3}^s} \|g^\epsilon\|_{H_{\gamma/2}^s} + \epsilon^{2-2s} \|f^\epsilon\|_{H_{l+3}^1} \|g^\epsilon\|_{H_{\gamma/2}^1}),$$

which implies, for any $\eta_3 > 0$,

$$(3.78) \quad |\langle \mathfrak{I}_2, g^\epsilon \rangle| - \eta_3 \|g^\epsilon\|_{\epsilon, \gamma/2}^2 \lesssim \frac{1}{\eta_3} \|f^\epsilon\|_{H_6^m}^2 \|f^\epsilon\|_{\epsilon, l+3}^2.$$

While for the case $1 \leq |\alpha_1| \leq |\alpha| - 1 \leq m - 1$, we similarly have, for any $\eta_3 > 0$,

$$(3.79) \quad |\langle \mathfrak{I}_2, g^\epsilon \rangle| - \eta_3 \|g^\epsilon\|_{\epsilon, \gamma/2}^2 \lesssim \frac{1}{\eta_3} (\|f^\epsilon\|_{H_6^1}^2 \|f^\epsilon\|_{H_{l+3}^m}^2 \mathbf{1}_{m \geq 2} + \|f^\epsilon\|_{H_6^{m-1}}^2 \|f^\epsilon\|_{H_{l+3}^{m-1}}^2).$$

Now choosing suitable η_1 in (3.76), η_2 in (3.77) and (3.78), and η_3 in (3.78) and (3.79), we have

$$\begin{aligned} (3.80) \quad &\frac{d}{dt} \|f^\epsilon\|_{H_l^m}^2 + \frac{C_1(f_0)}{2} \|f^\epsilon\|_{\epsilon, m, l+\gamma/2}^2 \\ &\lesssim C(\|f_0\|_{L_{2l+5}^1}, \|f_0\|_{L^2}) \|f^\epsilon\|_{H_{l+\gamma/2}^m}^2 \\ &\quad + C(C_1(f_0)) \{ \|f^\epsilon\|_{H_{2l+7}^m}^2 \|f^\epsilon\|_{\epsilon, l+3}^2 + \|f^\epsilon\|_{H_{2l+7}^1}^2 \|f^\epsilon\|_{H_{l+3}^m}^2 \mathbf{1}_{m \geq 2} + \|f^\epsilon\|_{H_{2l+7}^{m-1}}^2 \|f^\epsilon\|_{H_{l+3}^{m-1}}^2 \}. \end{aligned}$$

When $m = 1$, inequality (3.80) reduces to

$$\begin{aligned} &\frac{d}{dt} \|f^\epsilon\|_{H_l^1}^2 + \frac{C_1(f_0)}{2} \|f^\epsilon\|_{\epsilon, 1, l+\gamma/2}^2 \\ &\lesssim C(\|f_0\|_{L_{2l+5}^1}, \|f_0\|_{L^2}) \|f^\epsilon\|_{H_{l+\gamma/2}^1}^2 \\ &\quad + C(C_1(f_0)) \{ \|f^\epsilon\|_{H_{2l+7}^1}^2 \|f^\epsilon\|_{\epsilon, l+3}^2 + \|f^\epsilon\|_{L_{2l+7}^2}^2 \|f^\epsilon\|_{L_{l+3}^2}^2 \}. \end{aligned}$$

Remembering that

$$\|f^\epsilon\|_{\epsilon, l+3}^2 = \|f^\epsilon\|_{H_{l+3}^s}^2 + \epsilon^{2-2s} \|f^\epsilon\|_{H_{l+3}^1}^2,$$

by interpolation theory and the basic inequality (1.10), for any $\eta > 0$, we have

$$\begin{aligned} \|f^\epsilon\|_{H_{2l+7}^1}^2 &\leq \|f^\epsilon\|_{H_{l+\gamma/2}^{1+s}}^{2(1-s)} \|f^\epsilon\|_{H_{\psi(l)}^s}^{2s} \\ &\leq \eta \|f^\epsilon\|_{H_{l+\gamma/2}^{1+s}}^2 + s \left(\frac{\eta}{1-s} \right)^{-\frac{1-s}{s}} \|f^\epsilon\|_{H_{x(l)}^s}^2, \end{aligned}$$

where $x(l) = \frac{2l+7}{s} - \frac{1-s}{s}(l + \frac{\gamma}{2})$, and

$$\begin{aligned} \|f^\epsilon\|_{H_{2l+7}^1}^2 \|f^\epsilon\|_{H_{l+3}^1}^2 &\leq \|f^\epsilon\|_{H_{2l+7}^1}^4 \\ &\leq \|f^\epsilon\|_{H_{l+\gamma/2}^2}^{\frac{4-4s}{2-s}} \|f^\epsilon\|_{H_{\psi'(l)}^s}^{\frac{4}{2-s}} \\ &\leq \eta \|f^\epsilon\|_{H_{l+\gamma/2}^2}^2 + \frac{s}{2-s} \left(\frac{2\eta - s\eta}{2-2s} \right)^{-\frac{2-2s}{s}} \|f^\epsilon\|_{H_{\tilde{x}(l)}^s}^{4/s}, \end{aligned}$$

where $\tilde{x}(l) = (2l+7)(2-s) - (1-s)(l + \gamma/2) \leq x(l)$. Taking small enough η , we finally have

$$\frac{d}{dt} \|f^\epsilon\|_{H_l^1}^2 + \frac{C_1(f_0)}{4} \|f^\epsilon\|_{\epsilon,1,l+\gamma/2}^2 \lesssim C(\|f^\epsilon\|_{H_{x(l)}^s}, \|f_0\|_{L_{2l+5}^1}).$$

Then by Gronwall's inequality and the uniform upper bound (3.73) of H^s norm, we arrive at

$$\|f^\epsilon(t)\|_{H_l^1}^2 + \frac{C_1(f_0)}{4} \int_t^{t+1} \|f^\epsilon(r)\|_{\epsilon,1,l+\gamma/2}^2 dr \lesssim C(\|f_0\|_{L_{\phi(s,x(l))}^1}, \|f_0\|_{H_{x(l)}^s}, \|f_0\|_{H_l^1}).$$

Once again by interpolation theory, there holds

$$\|f_0\|_{H_{x(l)}^s} \lesssim \|f_0\|_{H_l^1} + \|f_0\|_{L_{y(l)}^1},$$

where $y(l) = \frac{3x(l)-(s+2)l}{1-s}$. By setting $\phi(1, l) = \max\{\phi(s, x(l)), y(l)\}$, we have

$$\|f^\epsilon(t)\|_{H_l^1}^2 + \frac{C_1(f_0)}{4} \int_t^{t+1} \|f^\epsilon(r)\|_{\epsilon,1,l+\gamma/2}^2 dr \lesssim C(\|f_0\|_{L_{\phi(1,l)}^1}, \|f_0\|_{H_l^1}).$$

When $m \geq 2$, $\|f^\epsilon\|_{H_{2l+7}^1}^2$ has uniform bound by assumption. According to the interpolation inequality and the basic inequality (1.10), one has

$$(3.81) \quad \|f^\epsilon\|_{H_{2l+7}^m}^2 \leq \eta \|f^\epsilon\|_{H_{l+\gamma/2}^{m+s}}^2 + \left(\frac{1+s}{s}\eta\right)^{-\frac{1}{s}} \|f^\epsilon\|_{H_{z(l)}^{m-1}}^2,$$

where $z(l) = 2l+7 + \frac{l+7}{s}$. With the fact $\|f^\epsilon\|_{\epsilon,l+3} \lesssim \|f^\epsilon\|_{H_{l+3}^1}$, we finally arrive at

$$\frac{d}{dt} \|f^\epsilon\|_{H_l^m}^2 + \frac{C_1(f_0)}{4} \|f^\epsilon\|_{\epsilon,m,l+\gamma/2}^2 \lesssim C(\|f_0\|_{L_{2l+5}^1}, \|f^\epsilon\|_{H_{z(l)}^{m-1}}).$$

Then by Gronwall's inequality and the assumed uniform bound of H^{m-1} norm,

$$\|f^\epsilon(t)\|_{H_l^m}^2 + \frac{C_1(f_0)}{4} \int_t^{t+1} \|f^\epsilon(r)\|_{\epsilon,m,l+\gamma/2}^2 dr \lesssim C(\|f_0\|_{L_{\phi(m-1,z(l))}^1}, \|f_0\|_{H_{z(l)}^{m-1}}, \|f_0\|_{H_l^m}).$$

By interpolation theory, there holds

$$\|f_0\|_{H_{z(l)}^{m-1}} \lesssim \|f_0\|_{H_l^m} + \|f_0\|_{L_{u(m,l)}^1},$$

where $u(m, l) = (m+2)z(l) - (m+1)l$. Now by setting $\phi(m, l) = \max\{u(m, l), \phi(m-1, z(l))\}$, we arrive at

$$(3.82) \quad \|f^\epsilon(t)\|_{H_l^m}^2 + \frac{C_1(f_0)}{4} \int_t^{t+1} \|f^\epsilon(r)\|_{\epsilon,m,l+\gamma/2}^2 dr \lesssim C(\|f_0\|_{L_{\phi(m,l)}^1}, \|f_0\|_{H_l^m}).$$

The proof of theorem 1.1 is complete now.

Remark 3.2. Since $L^1 \subset H^{-m}$ if $m > 3/2$, one can obtain lower weight requirement in the space L^1 . We use H^{-2} as one interpolation space just for a neat expression. For the same reason, we replace γ with 2.

4. ERROR ESTIMATES TO THE APPROXIMATION

In this section, we prove the last two theorems stated in section 1. We first give a proof to theorem 1.2.

Proof of Theorem 1.2: For each $0 < \epsilon \leq \sqrt{2}/2$, we define F_R^ϵ and Q_ϵ respectively as follows:

$$(4.1) \quad F_R^\epsilon = \frac{f^\epsilon - f}{\epsilon^{3-2s}},$$

$$(4.2) \quad Q_\epsilon = Q - Q^\epsilon.$$

Take the difference between equations (1.1) and (1.7), and divide both sides by ϵ^{3-2s} , we have

$$(4.3) \quad \partial_t F_R^\epsilon = \Upsilon(f^\epsilon) + Q(f^\epsilon, F_R^\epsilon) + Q(F_R^\epsilon, f)$$

where

$$(4.4) \quad \Upsilon(f^\epsilon) = \frac{1}{\epsilon} [Q_L(f^\epsilon, f^\epsilon) - \epsilon^{2s-2} Q_\epsilon(f^\epsilon, f^\epsilon)].$$

We now show that L_{2l}^1 norm of F_R^ϵ is bounded by the initial datum f_0 and time t . According to (4.3), we have

$$(4.5) \quad \begin{aligned} \frac{d}{dt} \|F_R^\epsilon\|_{L_{2l}^1} &= \langle \Upsilon(f^\epsilon) + Q(f^\epsilon, F_R^\epsilon) + Q(F_R^\epsilon, f), \text{sgn}(F_R^\epsilon) \langle v \rangle^{2l} \rangle \\ &\stackrel{\text{def}}{=} \mathfrak{I}_1 + \mathfrak{I}_2 + \mathfrak{I}_3. \end{aligned}$$

Thanks to lemma (7.1) in the Appendix of [9], we have

$$(4.6) \quad \mathfrak{I}_1 \leq C(7.1) \|f^\epsilon\|_{H_{2l+\gamma+12}^5}^2.$$

Now we deal with \mathfrak{I}_2 , note that

$$\begin{aligned} \mathfrak{I}_2 &= \int_{\mathbb{R}^6 \times SS^2} b(\cos \theta) |v - v_*|^\gamma f_*^\epsilon F_R^\epsilon (\text{sgn}(F_R^\epsilon(v')) \langle v' \rangle^{2l} - \text{sgn}(F_R^\epsilon(v)) \langle v \rangle^{2l}) d\sigma dv_* dv \\ &\leq \int_{\mathbb{R}^6 \times SS^2} b(\cos \theta) |v - v_*|^\gamma f_*^\epsilon |F_R^\epsilon| (\langle v' \rangle^{2l} - \langle v \rangle^{2l}) d\sigma dv_* dv \\ &\stackrel{\text{def}}{=} \mathfrak{I}_{2,1} + \mathfrak{I}_{2,2}, \end{aligned}$$

where

$$\mathfrak{I}_{2,1} = \int_{\mathbb{R}^6 \times SS^2} b(\cos \theta) |v - v_*|^\gamma f_*^\epsilon |F_R^\epsilon| (\langle v' \rangle^{2l} + \langle v_*' \rangle^{2l} - \langle v \rangle^{2l} - \langle v_* \rangle^{2l}) d\sigma dv_* dv,$$

and

$$\mathfrak{I}_{2,2} = - \int_{\mathbb{R}^6 \times SS^2} b(\cos \theta) |v - v_*|^\gamma f_*^\epsilon |F_R^\epsilon| (\langle v_*' \rangle^{2l} - \langle v_* \rangle^{2l}) d\sigma dv_* dv.$$

According to proposition 3.2, we have

$$(4.7) \quad \mathfrak{I}_{2,1} \leq -\frac{1}{4} A_2 (\|f^\epsilon\|_{L^1} \|F_R^\epsilon\|_{L_{2l+\gamma}^1} + \|f^\epsilon\|_{L_{2l+\gamma}^1} \|F_R^\epsilon\|_{L^1}) + 2^{2l+1} A_2 \|f^\epsilon\|_{L_{2l}^1} \|F_R^\epsilon\|_{L_{2l}^1}.$$

Now we turn to $\mathfrak{I}_{2,2}$. Recall that

$$\begin{aligned} \langle v_*' \rangle^{2l} - \langle v_* \rangle^{2l} &= (v_*' - v_*) \cdot (\nabla \langle \cdot \rangle^{2l})(v_*) \\ &\quad + \int_0^1 \frac{1-\kappa}{2} (v_*' - v_*) \otimes (v_*' - v_*) : (\nabla^2 \langle \cdot \rangle^{2l})(v(\kappa)) d\kappa, \end{aligned}$$

where $v(\kappa) = v_* + \kappa(v_*' - v_*) = v_* - \kappa(v' - v)$. By symmetry,

$$(4.8) \quad \int_{SS^2} b(\cos \theta) (v_*' - v_*) d\sigma = (v - v_*) \int_{SS^2} b(\cos \theta) \sin^2 \frac{\theta}{2} d\sigma.$$

Observe that the matrix $\nabla^2 \langle \cdot \rangle^{2l}$ is positive definite, we are only left with

$$(4.9) \quad \begin{aligned} \mathfrak{I}_{2,2} &\leq 2l \int_{\mathbb{R}^6 \times SS^2} b(\cos \theta) \sin^2 \frac{\theta}{2} |v - v_*|^{\gamma+1} \langle v_* \rangle^{2l-1} f_*^\epsilon |F_R^\epsilon| d\sigma dv_* dv \\ &\leq A_2 l \|f^\epsilon\|_{L_{2l+\gamma}^1} \|F_R^\epsilon\|_{L_{\gamma+1}^1}. \end{aligned}$$

Split \mathfrak{I}_3 into two parts:

$$\begin{aligned} \mathfrak{I}_3 &= \int_{\mathbb{R}^6 \times SS^2} b(\cos \theta) |v - v_*|^\gamma F_R^\epsilon(v_*) f(\text{sgn}(F_R^\epsilon(v')) \langle v' \rangle^{2l} - \text{sgn}(F_R^\epsilon(v)) \langle v \rangle^{2l}) d\sigma dv_* dv \\ &= \int_{\mathbb{R}^6 \times SS^2} B \mathbf{1}_{\theta \leq |v-v_*|^{-\alpha}} F_R^\epsilon(v_*) f(\text{sgn}(F_R^\epsilon(v')) \langle v' \rangle^{2l} - \text{sgn}(F_R^\epsilon(v)) \langle v \rangle^{2l}) d\sigma dv_* dv \\ &\quad + \int_{\mathbb{R}^6 \times SS^2} B \mathbf{1}_{\theta \geq |v-v_*|^{-\alpha}} F_R^\epsilon(v_*) f(\text{sgn}(F_R^\epsilon(v')) \langle v' \rangle^{2l} - \text{sgn}(F_R^\epsilon(v)) \langle v \rangle^{2l}) d\sigma dv_* dv \\ &\stackrel{\text{def}}{=} \mathfrak{I}_{3,1} + \mathfrak{I}_{3,2}, \end{aligned}$$

where $\alpha = \frac{\gamma+2}{2-2s}$.
For $\mathfrak{I}_{3,1}$, we have

$$\begin{aligned}\mathfrak{I}_{3,1} &= \int_{\mathbb{R}^6 \times SS^2} B \mathbf{1}_{\theta \leq |v-v_*|^{-\alpha}} F_R^\epsilon(v_*) (sgn(F_R^\epsilon(v')) f' \langle v' \rangle^{2l} - sgn(F_R^\epsilon(v)) f \langle v \rangle^{2l}) d\sigma dv_* dv \\ &\quad + \int_{\mathbb{R}^6 \times SS^2} b(\cos \theta) \mathbf{1}_{\theta \leq |v-v_*|^{-\alpha}} |v-v_*|^\gamma F_R^\epsilon(v_*) (f-f') sgn(F_R^\epsilon(v')) \langle v' \rangle^{2l} d\sigma dv_* dv \\ &\stackrel{\text{def}}{=} \mathfrak{I}_{3,1,1} + \mathfrak{I}_{3,1,2}.\end{aligned}$$

By cancellation lemma,

$$(4.10) \quad |\mathfrak{I}_{3,1,1}| \leq C(\text{cancel}) \|f\|_{L_{2l+\gamma}^1} \|F_R^\epsilon\|_{L_\gamma^1},$$

where $C(\text{cancel}) = 2^{\frac{5+\gamma}{2}} A_2$. For the term $\mathfrak{I}_{3,1,2}$, apply Taylor expansion:

$$f(v) - f(v') = (v-v') \cdot \nabla_v f(v') + \int_0^1 \frac{1-\kappa}{2} (v-v') \otimes (v-v') : \nabla_v^2 f(v(\kappa)) d\kappa,$$

where $v(\kappa) = v' + \kappa(v-v')$. For fixed v_* , it is easy to check

$$\int_{\mathbb{R}^3 \times SS^2} b(\cos \theta) \mathbf{1}_{\theta \leq |v-v_*|^{-\alpha}} |v-v_*|^\gamma (v-v') \cdot \nabla_v f(v') sgn(F_R^\epsilon(v')) \langle v' \rangle^{2l} d\sigma dv = 0.$$

Thus we are only left with

$$(4.11) \quad \begin{aligned}|\mathfrak{I}_{3,1,2}| &\leq \int_0^1 \int_{\mathbb{R}^6 \times SS^2} \frac{1-\kappa}{2} b(\cos \theta) \sin^2 \frac{\theta}{2} \mathbf{1}_{\theta \leq |v-v_*|^{-\alpha}} |v-v_*|^{\gamma+2} \\ &\quad \times F_R^\epsilon(v_*) |\nabla_v^2 f(v(\kappa))| \langle v' \rangle^{2l} d\kappa d\sigma dv_* dv.\end{aligned}$$

Set $u = v' + \kappa(v-v')$, then we have

$$\begin{aligned}\langle v' \rangle^2 &= 1 + |v'|^2 = 1 + |v' + \kappa(v-v') - \kappa(v-v')|^2 \\ &\leq 1 + 2|u|^2 + 2\kappa^2 |v-v'|^2 \leq 2\langle u \rangle^2 + 2\kappa^2 |u-v_*|^2,\end{aligned}$$

and

$$\langle v' \rangle^{2l} \leq 2^{2l-1} \langle u \rangle^{2l} + 2^{2l-1} \kappa^{2l} |u-v_*|^{2l} \leq 2^{2l} \langle u \rangle^{2l} \langle v_* \rangle^{2l}.$$

By the change of variable: $v \rightarrow u$, the Jacobian matrix is

$$\frac{du}{dv} = \frac{1+k}{2} \left(I + \frac{1-k}{1+k} \frac{v-v_*}{|v-v_*|} \otimes \sigma \right),$$

with its Jacobian

$$\left| \frac{du}{dv} \right| = \frac{(1+k)^3}{8} \left(1 + \frac{1-k}{1+k} \frac{v-v_*}{|v-v_*|} \cdot \sigma \right) \geq \frac{1}{8}.$$

Thanks to $|u-v_*| \leq |v-v_*| \leq \sqrt{2}|u-v_*|$, we obtain

$$(4.12) \quad \begin{aligned}|\mathfrak{I}_{3,1,2}| &\leq 2^{2l+3} \pi K \int_{\mathbb{R}^6} \int_0^{|u-v_*|^{-\alpha} \wedge \pi/2} \theta^{1-2s} |u-v_*|^{\gamma+2} \\ &\quad \times F_R^\epsilon(v_*) |\nabla_v^2 f(u)| \langle u \rangle^{2l} \langle v_* \rangle^{2l} d\theta dv_* du. \\ &\leq 2^{2l+2} \frac{\pi K}{1-s} \|\nabla_v^2 f\|_{L_{2l}^1} \|F_R^\epsilon\|_{L_{2l}^1} \\ &\leq 2^{2l+\frac{5}{2}} \frac{\pi^2 K}{1-s} \|f\|_{H_{2l+2}^2} \|F_R^\epsilon\|_{L_{2l}^1},\end{aligned}$$

where we have used the fact $\|\langle \cdot \rangle^{-2}\|_{L^2} \leq \sqrt{2}\pi$.

Now we turn to $\mathfrak{I}_{3,2}$. Note that

$$\begin{aligned}
\mathfrak{I}_{3,2} &\leq \int_{\mathbb{R}^6 \times SS^2} b(\cos \theta) \mathbf{1}_{\theta \geq |v-v_*|^{-\alpha}} |v-v_*|^\gamma |F_R^\epsilon(v_*)| f(\langle v' \rangle^{2l} + \langle v \rangle^{2l}) d\sigma dv_* dv \\
&= \int_{\mathbb{R}^6 \times SS^2} b(\cos \theta) \mathbf{1}_{\theta \geq |v-v_*|^{-\alpha}} |v-v_*|^\gamma |F_R^\epsilon(v_*)| f(\langle v' \rangle^{2l} - \langle v \rangle^{2l}) d\sigma dv_* dv \\
&\quad + 2 \int_{\mathbb{R}^6 \times SS^2} b(\cos \theta) \mathbf{1}_{\theta \geq |v-v_*|^{-\alpha}} |v-v_*|^\gamma |F_R^\epsilon(v_*)| f\langle v \rangle^{2l} d\sigma dv_* dv \\
&\stackrel{\text{def}}{=} \mathfrak{I}_{3,2,1} + \mathfrak{I}_{3,2,2}.
\end{aligned}$$

First look at the term $\mathfrak{I}_{3,2,1}$. Recall that $j = \frac{u-(u \cdot n)n}{|u-(u \cdot n)n|}$ in lemma 3.1, then we have $j \cdot n = 0$, and thus

$$\int_{SS^2} b(\cos \theta) \mathbf{1}_{\theta \geq |v-v_*|^{-\alpha}} (E(\theta))^{p-1} h(j \cdot \omega) \sin \theta d\sigma = 0.$$

Applying proposition 3.1 and the above equality, we obtain

$$\begin{aligned}
\mathfrak{I}_{3,2,1} &\leq \int_{\mathbb{R}^6 \times SS^2} b(\cos \theta) \sin^{2l} \frac{\theta}{2} |v-v_*|^\gamma |F_R^\epsilon(v_*)| f\langle v_* \rangle^{2l} d\sigma dv_* dv \\
&\quad + c_l \int_{\mathbb{R}^6 \times SS^2} b(\cos \theta) \sin^2 \theta |v-v_*|^\gamma |F_R^\epsilon(v_*)| f\langle v_* \rangle^{2l-2} \langle v \rangle^{2l-2} d\sigma dv_* dv \\
&\stackrel{\text{def}}{=} \mathfrak{I}_{3,2,1,1} + \mathfrak{I}_{3,2,1,2},
\end{aligned}$$

where $c_l = 2^{l-3}(l(l-1)+4)$. Thanks to the following fact:

$$\begin{aligned}
\int_{SS^2} b(\cos \theta) \sin^{2l} \frac{\theta}{2} d\sigma &\leq 2\pi K \int_0^{\pi/2} \theta^{-1-2s} \sin^{2l} \frac{\theta}{2} d\theta \\
&= 2^{1-2s} \pi K \int_0^{\pi/4} \eta^{-1-2s} \sin^{2l} \eta d\eta \\
&\leq \frac{2^{-2s} \pi K}{l-s} \left(\frac{\pi}{4}\right)^{2l-2s},
\end{aligned}$$

and $|v-v_*|^\gamma \leq 2(\langle v \rangle^\gamma + \langle v_* \rangle^\gamma)$, we have

$$(4.13) \quad \mathfrak{I}_{3,2,1,1} \leq \frac{2^{1-2s} \pi K}{l-s} \left(\frac{\pi}{4}\right)^{2l-2s} (\|f\|_{L^1} \|F_R^\epsilon\|_{L_{2l+\gamma}^1} + \|f\|_{L_\gamma^1} \|F_R^\epsilon\|_{L_{2l}^1}).$$

Due to $|v-v_*|^\gamma \leq \langle v \rangle^2 \langle v_* \rangle^2$, we obtain

$$(4.14) \quad \mathfrak{I}_{3,2,1,2} \leq c_l A_2 \|f\|_{L_{2l}^1} \|F_R^\epsilon\|_{L_{2l}^1}.$$

For the term $\mathfrak{I}_{3,2,2}$, we have

$$\begin{aligned}
(4.15) \quad |\mathfrak{I}_{3,2,2}| &\leq 4\pi K \int_{\mathbb{R}^6} \int_{|v-v_*|^{-\alpha} \wedge \pi/2}^{\pi/2} \theta^{-1-2s} |v-v_*|^\gamma |F_R^\epsilon(v_*)| f\langle v \rangle^{2l} d\theta dv_* dv \\
&\leq \frac{2\pi K}{s} \|f\|_{L_{4l}^1} \|F_R^\epsilon\|_{L_{2l}^1},
\end{aligned}$$

provided $2\alpha s + \gamma \leq 2l$.

Patch the above inequalities (4.6), (4.7), (4.9), (4.10), and (4.12)-(4.15), for those l such that $\frac{2^{1-2s} \pi K}{l-s} \left(\frac{\pi}{4}\right)^{2l-2s} \leq \frac{A_2}{8}$, we have the following desired result:

$$\begin{aligned}
\frac{d}{dt} \|F_R^\epsilon\|_{L_{2l}^1} &\leq -\frac{A_2}{8} \|f_0\|_{L^1} \|F_R^\epsilon\|_{L_{2l+\gamma}^1} + C(7.1) \|f^\epsilon\|_{H_{2l+\gamma+12}^5}^2 \\
&\quad + \{2^{2l+1} A_2 \|f^\epsilon\|_{L_{2l}^1} + A_2 l \|f^\epsilon\|_{L_{2l+\gamma}^1} + C(\text{cancel}) \|f\|_{L_{2l+\gamma}^1} \\
&\quad + 2^{2l+\frac{1}{2}} \frac{\pi^2 K}{1-s} \|f\|_{H_{2l+2}^2} + \frac{A_2}{8} \|f\|_{L_\gamma^1} \\
&\quad + c_l A_2 \|f\|_{L_{2l}^1} + \frac{2\pi K}{s} \|f\|_{L_{4l}^1} \} \|F_R^\epsilon\|_{L_{2l}^1},
\end{aligned}$$

where we have used the mass conservation property: $\|f_t\|_{L^1} = \|f_t^\epsilon\|_{L^1} = \|f_0\|_{L^1}$. The propagation of L and H norms of f and f^ϵ allows us to conclude:

$$(4.16) \quad \begin{aligned} \frac{d}{dt} \|F_R^\epsilon\|_{L_{2l}^1} &\leq -\frac{A_2}{8} \|f_0\|_{L^1} \|F_R^\epsilon\|_{L_{2l+\gamma}^1} + C(\|f_0\|_{L_{\phi(5,2l+\gamma+12)}^1}, \|f_0\|_{H_{2l+\gamma+12}^5}) \\ &\quad + C(\|f_0\|_{L_{\max\{41, \phi(2,2l+2)\}}^1}, \|f_0\|_{H_{2l+2}^2}) \|F_R^\epsilon\|_{L_{2l}^1}. \end{aligned}$$

Applying Gronwall's inequality (1.11) with $a = C(\|f_0\|_{L_{\phi(5,2l+\gamma+12)}^1}, \|f_0\|_{H_{2l+\gamma+12}^5})$ and $b = C(\|f_0\|_{L_{\max\{41, \phi(2,2l+2)\}}^1}, \|f_0\|_{H_{2l+2}^2})$, we have

$$\|F_R^\epsilon(t)\|_{L_{2l}^1} \leq \frac{a}{b} (e^{bt} - 1) \stackrel{\text{def}}{=} C(f_0, t).$$

We now prove theorem 1.3 in the rest of this section.

Proof of Theorem 1.3:

Step 1: (Case $N = 0$)

Taking the difference between equations (1.1) and (1.7), and dividing both sides by ϵ^{3-2s} , we have

$$\partial_t F_R^\epsilon = \Upsilon(f) + M^\epsilon(f^\epsilon, F_R^\epsilon) + M^\epsilon(F_R^\epsilon, f)$$

Then we have

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|F_R^\epsilon\|_{L_l^2}^2 \right) &= \langle \Upsilon(f) + M^\epsilon(f^\epsilon, F_R^\epsilon) + M^\epsilon(F_R^\epsilon, f), F_R^\epsilon \langle v \rangle^{2l} \rangle \\ &\stackrel{\text{def}}{=} \mathfrak{I}_1 + \mathfrak{I}_2 + \mathfrak{I}_3 \end{aligned}$$

Thanks to lemma 7.1 in the Appendix of [9], we have

$$\mathfrak{I}_1 \lesssim \|f\|_{H_{l+\gamma+10}^5}^2 \|F_R^\epsilon\|_{L_l^2}^2,$$

which implies, for any $\eta > 0$,

$$(4.17) \quad \mathfrak{I}_1 - \eta \|F_R^\epsilon\|_{L^2}^2 \lesssim \frac{1}{\eta} \|f\|_{H_{l+\gamma+10}^5}^4.$$

Splitting \mathfrak{I}_2 into two terms

$$\begin{aligned} \mathfrak{I}_2 &= \langle M^\epsilon(f^\epsilon, F_R^\epsilon \langle v \rangle^l), F_R^\epsilon \langle v \rangle^l \rangle + \{ \langle M^\epsilon(f^\epsilon, F_R^\epsilon) \langle v \rangle^l - M^\epsilon(f^\epsilon, F_R^\epsilon \langle v \rangle^l), F_R^\epsilon \langle v \rangle^l \rangle \} \\ &\stackrel{\text{def}}{=} \mathfrak{I}_{2,1} + \mathfrak{I}_{2,2}. \end{aligned}$$

By coercivity estimate (2.5), we have

$$(4.18) \quad \mathfrak{I}_{2,1} \leq -C_1(f_0) \|F_R^\epsilon\|_{\epsilon, l+\gamma/2}^2 + C_2(f_0) \|F_R^\epsilon\|_{L_{l+\gamma/2}^2}^2.$$

By commutator estimate (2.9) with $N_2 = l + \gamma/2$, $N_3 = \gamma/2$, we have,

$$\mathfrak{I}_{2,2} \lesssim \|f^\epsilon\|_{L_{2l+5}^1} \|F_R^\epsilon\|_{\epsilon, l+\gamma/2} \|F_R^\epsilon\|_{L_{l+\gamma/2}^2},$$

which implies, for any $\eta > 0$,

$$(4.19) \quad \mathfrak{I}_{2,2} - \eta \|F_R^\epsilon\|_{\epsilon, l+\gamma/2}^2 \lesssim \frac{1}{\eta} \|f^\epsilon\|_{L_{2l+5}^1}^2 \|F_R^\epsilon\|_{L_{l+\gamma/2}^2}^2.$$

Splitting \mathfrak{I}_3 into two terms

$$\begin{aligned} \mathfrak{I}_3 &= \langle M^\epsilon(F_R^\epsilon, f \langle v \rangle^l), F_R^\epsilon \langle v \rangle^l \rangle_v + \{ \langle M^\epsilon(F_R^\epsilon, f) \langle v \rangle^l - M^\epsilon(F_R^\epsilon, f \langle v \rangle^l), F_R^\epsilon \langle v \rangle^l \rangle_v \} \\ &\stackrel{\text{def}}{=} \mathfrak{I}_{3,1} + \mathfrak{I}_{3,2}. \end{aligned}$$

Applying upper bound estimate (2.1) with $w_1 = \gamma/2 + 2$, $w_2 = \gamma/2$, we have

$$\mathfrak{I}_{3,1} \lesssim \|F_R^\epsilon\|_{L_{\gamma+2}^1} \|f\|_{H_{l+3}^1} \|F_R^\epsilon\|_{\epsilon, l+\gamma/2},$$

which implies, for any $\eta > 0$,

$$(4.20) \quad \mathfrak{I}_{3,1} - \eta \|F_R^\epsilon\|_{\epsilon, l+\gamma/2}^2 \lesssim \frac{1}{\eta} \|F_R^\epsilon\|_{L_{\gamma+2}^1}^2 \|f\|_{H_{l+3}^1}^2.$$

By commutator estimate (2.9), we have

$$\mathfrak{I}_{3,2} \lesssim \|F_R^\epsilon\|_{L_{2l+5}^1} \|f\|_{H_{l+\gamma/2}^1} \|F_R^\epsilon\|_{L_{l+\gamma/2}^2}$$

which implies, for any $\eta > 0$,

$$(4.21) \quad \mathfrak{I}_{3,2} - \eta \|F_R^\epsilon\|_{L_{l+\gamma/2}^2}^2 \lesssim \frac{1}{\eta} \|F_R^\epsilon\|_{L_{2l+5}^1}^2 \|f\|_{H_{l+\gamma/2}^1}^2.$$

Now setting $\eta = \frac{C_1(f_0)}{8}$ in (4.17), (4.19), (4.20), (4.21), and combining with (4.18), we have

$$(4.22) \quad \begin{aligned} & \frac{d}{dt} \|F_R^\epsilon\|_{L_l^2}^2 + C_1(f_0) \|F_R^\epsilon\|_{\epsilon, l+\gamma/2}^2 \\ & \lesssim \{C_2(f_0) + \frac{1}{\eta} \|f^\epsilon\|_{L_{2l+5}^1}^2\} \|F_R^\epsilon\|_{L_{l+\gamma/2}^2}^2 + \frac{1}{\eta} \|f\|_{H_{l+\gamma+10}^5}^4 \\ & \quad + \frac{1}{\eta} \|F_R^\epsilon\|_{L_{\gamma+2}^1}^2 \|f\|_{H_{l+3}^1}^2 + \frac{1}{\eta} \|F_R^\epsilon\|_{L_{2l+5}^1}^2 \|f\|_{H_{l+\gamma/2}^1}^2 \end{aligned}$$

Now choosing $\lambda = \frac{C_1(f_0)}{2} (C_2(f_0) + \frac{1}{\eta} \|f^\epsilon\|_{L_{2l+5}^1}^2)^{-1}$ in (3.16), we have

$$(4.23) \quad \begin{aligned} & \frac{d}{dt} \|F_R^\epsilon\|_{L_l^2}^2 + \frac{C_1(f_0)}{2} \|F_R^\epsilon\|_{\epsilon, l+\gamma/2}^2 \\ & \lesssim \{C_2(f_0) + \frac{1}{\eta} \|f^\epsilon\|_{L_{2l+5}^1}^2\} \lambda^{-\frac{3}{2s}} \|F_R^\epsilon\|_{L_{l+\gamma/2}^1}^2 + \frac{1}{\eta} \|f\|_{H_{l+\gamma+10}^5}^4 \\ & \quad + \frac{1}{\eta} \|F_R^\epsilon\|_{L_{\gamma+2}^1}^2 \|f\|_{H_{l+3}^1}^2 + \frac{1}{\eta} \|F_R^\epsilon\|_{L_{2l+5}^1}^2 \|f\|_{H_{l+\gamma/2}^1}^2 \end{aligned}$$

According to theorem 1.2, we have

$$\|F_R^\epsilon(t)\|_{L_{2l+5}^1} \leq C(\|f_0\|_{L_{\phi(5, 2l+\gamma+17)}^1}, \|f_0\|_{H_{2l+\gamma+17}^5}, t).$$

The other terms of the right hand side of (4.23) are also bounded by some lower order or lower weight norm of initial datum f_0 , thus we arrive at

$$\|F_R^\epsilon(t)\|_{L_l^2} \leq C(\|f_0\|_{L_{\phi(5, 2l+\gamma+17)}^1}, \|f_0\|_{H_{2l+\gamma+17}^5}, t).$$

We remark that the dependence on t is also at most exponential.

Step 2: (Case $N \geq 1$)

Suppose inequality (1.21) holds true for all $N \leq m-1$, we now prove that it is also valid for $N = m$.

Let $g_{\alpha,l}^\epsilon = \langle v \rangle^l \partial_v^\alpha F_R^\epsilon$ with $|\alpha| \leq m$, then $g_{\alpha,l}^\epsilon$ solves

$$\partial_t g_{\alpha,l}^\epsilon = \sum_{\alpha_1 + \alpha_2 = \alpha} \binom{\alpha}{\alpha_1} [M^\epsilon(\partial_v^{\alpha_1} f^\epsilon, \partial_v^{\alpha_2} F_R^\epsilon) + M^\epsilon(\partial_v^{\alpha_1} F_R^\epsilon, \partial_v^{\alpha_2} f) + \Upsilon(\partial_v^{\alpha_1} f, \partial_v^{\alpha_2} f)] \langle v \rangle^l$$

Therefore we have

$$(4.24) \quad \begin{aligned} \frac{d}{dt} \left(\frac{1}{2} \|g_{\alpha,l}^\epsilon\|_{L^2}^2 \right) &= \langle \partial_t g_{\alpha,l}^\epsilon, g_{\alpha,l}^\epsilon \rangle \\ &= \sum_{\alpha_1 + \alpha_2 = \alpha} \binom{\alpha}{\alpha_1} \{ \langle M^\epsilon(\partial_v^{\alpha_1} f^\epsilon, \partial_v^{\alpha_2} F_R^\epsilon) \rangle^l, g_{\alpha,l}^\epsilon \rangle \\ & \quad + \langle M^\epsilon(\partial_v^{\alpha_1} F_R^\epsilon, \partial_v^{\alpha_2} f) \rangle^l, g_{\alpha,l}^\epsilon \rangle \\ & \quad + \langle \Upsilon(\partial_v^{\alpha_1} f, \partial_v^{\alpha_2} f) \rangle^l, g_{\alpha,l}^\epsilon \rangle \} \\ &\stackrel{\text{def}}{=} \sum_{\alpha_1 + \alpha_2 = \alpha} \binom{\alpha}{\alpha_1} \{ \mathfrak{I}_1(\alpha_1, \alpha_2) + \mathfrak{I}_2(\alpha_1, \alpha_2) + \mathfrak{I}_3(\alpha_1, \alpha_2) \}. \end{aligned}$$

Again by lemma 7.1 in the Appendix of [9], we have

$$\langle \Upsilon(\partial_v^{\alpha_1} f, \partial_v^{\alpha_2} f) \rangle^l, g_{\alpha,l}^\epsilon \rangle \lesssim \|f\|_{H_{l+\gamma+10}^{m+5}}^2 \|g_{\alpha,l}^\epsilon\|_{L^2},$$

which implies, for any $\eta > 0$,

$$(4.25) \quad \langle \Upsilon(\partial_v^{\alpha_1} f, \partial_v^{\alpha_2} f) \rangle^l, g_{\alpha,l}^\epsilon \rangle - \eta \|g_{\alpha,l}^\epsilon\|_{L^2} \lesssim \frac{1}{\eta} \|f\|_{H_{l+\gamma+10}^{m+5}}^4.$$

Splitting $\mathfrak{J}_1(\alpha_1, \alpha_2)$ into two terms, we have

$$\begin{aligned} \mathfrak{J}_1(\alpha_1, \alpha_2) &= \langle M^\epsilon(\partial_v^{\alpha_1} f^\epsilon, \langle v \rangle^l \partial_v^{\alpha_2} F_R^\epsilon), g_{\alpha, l}^\epsilon \rangle \\ &\quad + \{ \langle M^\epsilon(\partial_v^{\alpha_1} f^\epsilon, \partial_v^{\alpha_2} F_R^\epsilon) \langle v \rangle^l, g_{\alpha, l}^\epsilon \rangle - \langle M^\epsilon(\partial_v^{\alpha_1} f^\epsilon, \langle v \rangle^l \partial_v^{\alpha_2} F_R^\epsilon), g_{\alpha, l}^\epsilon \rangle \} \\ &\stackrel{\text{def}}{=} \mathfrak{J}_{1,1}(\alpha_1, \alpha_2) + \mathfrak{J}_{1,2}(\alpha_1, \alpha_2). \end{aligned}$$

By coercivity estimate (2.5), we have

$$(4.26) \quad \mathfrak{J}_{1,1}(0, \alpha) \leq -C_1(f_0) \|g_{\alpha, l}^\epsilon\|_{\epsilon, \gamma/2}^2 + C_2(f_0) \|g_{\alpha, l}^\epsilon\|_{L_{\gamma/2}^2}^2.$$

For $1 \leq |\alpha_1| \leq |\alpha| \leq m$, by upper bound estimate (2.1) with $w_1 = \gamma/2 + 2, w_2 = \gamma/2$, we have

$$\begin{aligned} \mathfrak{J}_{1,1}(\alpha_1, \alpha_2) &\lesssim \|\partial_v^{\alpha_1} f^\epsilon\|_{L_4^1} \|\partial_v^{\alpha_2} F_R^\epsilon\|_{H_{l+\gamma/2+2}^1} \|g_{\alpha, l}^\epsilon\|_{\epsilon, \gamma/2} \\ &\lesssim \|f^\epsilon\|_{H_6^m} \|F_R^\epsilon\|_{H_{l+\gamma/2+2}^m} \|g_{\alpha, l}^\epsilon\|_{\epsilon, \gamma/2}, \end{aligned}$$

which implies, for any $\eta_1 > 0$,

$$(4.27) \quad \mathfrak{J}_{1,1}(\alpha_1, \alpha_2) - \eta_1 \|g_{\alpha, l}^\epsilon\|_{\epsilon, \gamma/2}^2 \lesssim \frac{1}{\eta_1} \|f^\epsilon\|_{H_6^m}^2 \|F_R^\epsilon\|_{H_{l+\gamma/2+2}^m}^2.$$

By commutator estimates (2.9) with $N_2 = l + \gamma/2, N_3 = \gamma/2$, we have

$$\mathfrak{J}_{1,2}(\alpha_1, \alpha_2) \lesssim \|f^\epsilon\|_{H_{2l+7}^m} \|\partial_v^{\alpha_2} F_R^\epsilon\|_{\epsilon, l+\gamma/2} \|g_{\alpha, l}^\epsilon\|_{L_{\gamma/2}^2},$$

which implies, for any $\eta_2 > 0$,

$$(4.28) \quad \mathfrak{J}_{1,2}(\alpha_1, \alpha_2) - \eta_2 \|\partial_v^{\alpha_2} F_R^\epsilon\|_{\epsilon, l+\gamma/2}^2 \lesssim \frac{1}{\eta_2} \|f^\epsilon\|_{H_{2l+7}^m}^2 \|g_{\alpha, l}^\epsilon\|_{L_{\gamma/2}^2}^2.$$

Splitting $\mathfrak{J}_2(\alpha_1, \alpha_2)$ into two terms, we have

$$\begin{aligned} \mathfrak{J}_2(\alpha_1, \alpha_2) &= \langle M^\epsilon(\partial_v^{\alpha_1} F_R^\epsilon, \langle v \rangle^l \partial_v^{\alpha_2} f), g_{\alpha, l}^\epsilon \rangle \\ &\quad + \{ \langle M^\epsilon(\partial_v^{\alpha_1} F_R^\epsilon, \partial_v^{\alpha_2} f) \langle v \rangle^l, g_{\alpha, l}^\epsilon \rangle - \langle M^\epsilon(\partial_v^{\alpha_1} F_R^\epsilon, \langle v \rangle^l \partial_v^{\alpha_2} f), g_{\alpha, l}^\epsilon \rangle \} \\ &\stackrel{\text{def}}{=} \mathfrak{J}_{2,1}(\alpha_1, \alpha_2) + \mathfrak{J}_{2,2}(\alpha_1, \alpha_2). \end{aligned}$$

Applying upper bound estimate (2.1) with $w_1 = \gamma/2 + 2, w_2 = \gamma/2$, we may have

$$\begin{aligned} \mathfrak{J}_{2,1}(\alpha_1, \alpha_2) &\lesssim \|\partial_v^{\alpha_1} F_R^\epsilon\|_{L_4^1} \|\partial_v^{\alpha_2} f\|_{H_{l+\gamma/2+2}^1} \|g_{\alpha, l}^\epsilon\|_{\epsilon, \gamma/2} \\ &\lesssim \|F_R^\epsilon\|_{H_6^m} \|f\|_{H_{l+\gamma/2+2}^{m+1}} \|g_{\alpha, l}^\epsilon\|_{\epsilon, \gamma/2}, \end{aligned}$$

which implies, for any $\eta_1 > 0$,

$$(4.29) \quad \mathfrak{J}_{2,1}(\alpha_1, \alpha_2) - \eta_1 \|g_{\alpha, l}^\epsilon\|_{\epsilon, \gamma/2}^2 \lesssim \frac{1}{\eta_1} \|f\|_{H_{l+\gamma/2+2}^{m+1}}^2 \|F_R^\epsilon\|_{H_6^m}^2.$$

By commutator estimate (2.9) with $N_2 = l + \gamma/2, N_3 = \gamma/2$, we have

$$\mathfrak{J}_{2,2}(\alpha_1, \alpha_2) \lesssim \|F_R^\epsilon\|_{H_{2l+7}^m} \|f\|_{H_{l+\gamma/2}^{m+1}} \|g_{\alpha, l}^\epsilon\|_{L_{\gamma/2}^2},$$

which implies, for any $\eta_1 > 0$,

$$(4.30) \quad \mathfrak{J}_{2,2}(\alpha_1, \alpha_2) - \eta_1 \|g_{\alpha, l}^\epsilon\|_{L_{\gamma/2}^2}^2 \lesssim \frac{1}{\eta_1} \|f\|_{H_{l+\gamma/2}^{m+1}}^2 \|F_R^\epsilon\|_{H_{2l+7}^m}^2.$$

Patching all together (4.25), (4.26), (4.27), (4.28), (4.29), (4.30), and taking η_1 small enough, we arrive at

$$\begin{aligned} (4.31) \quad &\frac{d}{dt} \left(\frac{1}{2} \|g_{\alpha, l}^\epsilon\|_{L^2}^2 \right) + \frac{3C_1(f_0)}{4} \|g_{\alpha, l}^\epsilon\|_{\epsilon, \gamma/2}^2 - \eta_2 \|F_R^\epsilon\|_{\epsilon, m, l+\gamma/2}^2 \\ &\lesssim \|f\|_{H_{l+\gamma+10}^{m+5}}^4 + (C_2(f_0) + \frac{1}{\eta_2} \|f^\epsilon\|_{H_{2l+7}^m}^2) \|g_{\alpha, l}^\epsilon\|_{L_{\gamma/2}^2}^2 \\ &\quad + \frac{1}{\eta_1} (\|f^\epsilon\|_{H_6^m}^2 + \|f\|_{H_{l+\gamma/2+2}^{m+1}}^2 + \|f\|_{H_{l+\gamma/2}^{m+1}}^2) \|F_R^\epsilon\|_{H_{2l+7}^m}^2. \end{aligned}$$

Let $a(m) = \sum_{r=0}^m \binom{r+2}{r}$. Summing over $|\alpha| \leq m$, by taking $\eta_2 = \frac{C_1(f_0)}{4} \frac{1}{a(m)}$, we have

$$(4.32) \quad \begin{aligned} & \frac{d}{dt} \left(\frac{1}{2} \|F_R^\epsilon\|_{H_l^m}^2 \right) + \frac{C_1(f_0)}{2} \|F_R^\epsilon\|_{\epsilon, m, l+\gamma/2}^2 \\ & \lesssim \|f\|_{H_{l+\gamma+10}^{m+5}}^4 + (C_2(f_0) + \frac{1}{\eta_2} \|f^\epsilon\|_{H_{2l+7}^m}^2) \|F_R^\epsilon\|_{H_{l+\gamma/2}^m}^2 \\ & \quad + \frac{1}{\eta_1} (\|f^\epsilon\|_{H_6^m}^2 + \|f\|_{H_{l+\gamma/2+2}^{m+1}}^2 + \|f\|_{H_{l+\gamma/2}^{m+1}}^2) \|F_R^\epsilon\|_{H_{2l+7}^m}^2. \end{aligned}$$

Thanks to (3.81), we may conclude

$$\frac{d}{dt} \|F_R^\epsilon\|_{H_l^m}^2 + \frac{C_1(f_0)}{2} \|F_R^\epsilon\|_{\epsilon, m, l+\gamma/2}^2 \lesssim C(\|f\|_{H_{l+\gamma+10}^{m+5}}, \|F_R^\epsilon\|_{H_{z(l)}^{m-1}}, \|f_0\|_{L \log L}, \|f_0\|_{L_1^1}).$$

By theorem 1.1 and remark 1.1, for any $t \geq 0$, we have

$$\|f(t)\|_{H_{l+\gamma+10}^{m+5}} \lesssim C(\|f_0\|_{L_{\phi(m+5, l+\gamma+10)}^1}, \|f_0\|_{H_{l+\gamma+10}^{m+5}}).$$

By assumption, there holds

$$\|F_R^\epsilon(t)\|_{H_{z(l)}^{m-1}} \lesssim C(\|f_0\|_{L_{\varphi(m-1, z(l))}^1}, \|f_0\|_{H_{\psi(m-1, z(l))}^{m+4}}, t).$$

On the other hand, by interpolation, we have

$$\|f_0\|_{H_{\psi(m-1, z(l))}^{m+4}} \lesssim \|f_0\|_{H_{l+\gamma+10}^{m+5}} + \|f_0\|_{L_{\rho(m, l)}^1},$$

where $\rho(m, l) = (m+7)\psi(m-1, z(l)) - (m+6)(l+\gamma+10)$. By defining $\varphi(m, l) = \max\{\varphi(m-1, z(l)), \rho(m, l)\}$, we have

$$\|F_R^\epsilon(t)\|_{H_l^m}^2 \leq C(\|f_0\|_{L_{\varphi(m, l)}^1}, \|f_0\|_{H_{l+\gamma+10}^{m+5}}, t).$$

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REFERENCES

- [1] R. Alexandre, L. Desvillettes, C. Villani, and B. Wennberg, Entropy dissipation and long-range interactions, *Arch. Ration. Mech. Anal.* 152 (2000), no. 4, 327-355.
- [2] R. Alexandre, Y. Morimoto, S. Ukai, C.-J. Xu, and T. Yang, Smoothing effect of weak solutions for the spatially homogeneous Boltzmann equation without angular cutoff, *Kyoto J. Math.* 52 (2012), no. 3, 433-463.
- [3] L. Arkeryd, On the Boltzmann equation, *Arch. Rational Mech. Anal.* 45 (1972), 1-34.
- [4] Y. Chen and L. He, Smoothing estimates for Boltzmann equation with full-range interactions I: spatially homogeneous case, *Arch. Ration. Mech. Anal.* 201 (2011), no. 2, 501-548.
- [5] L. Desvillettes, On asymptotics of the Boltzmann equation when the collisions become grazing, *Transp. Theory Stat. Phys.* 21(3) (1992), 259-276.
- [6] L. Desvillettes, C. Villani, On the spatially homogeneous Landau equation for hard potentials part i: existence, uniqueness and smoothness, *Comm. Partial Differential Equations* 25.1-2 (2000), 179-259.
- [7] N. Fournier and D. Godinho, Asymptotic of grazing collisions and particle approximation for the Kac equation without cutoff, *Commun. Math. Phys.* 316.2 (2012), 307-344.
- [8] N. Fournier and A. Guillin, From a Kac-like particle system to the Landau equation for hard potentials and Maxwell molecules, *arXiv preprint arXiv:1510.01123*, (2015).
- [9] L. He, Asymptotic analysis of the spatially homogeneous Boltzmann equation: grazing collisions limit, *J. Stat. Phys.* 155.1 (2014), 151-210.
- [10] L. He, Well-posedness of spatially homogeneous Boltzmann equation with full-range interaction, *Commun. Math. Phys.* 312 (2012), 447-476.
- [11] L. He, Sharp bounds for Boltzmann and Landau collision operators, *arXiv:1604.06981*, (2016).
- [12] L.-B. He and J.-C. Jiang, On the Cauchy problem for inhomogeneous Boltzmann equations with Hard potentials: Well-posedness and Global stability, in preparation.
- [13] Z. Huo, Y. Morimoto, S. Ukai and T. Yang, Regularity of solutions for spatially homogeneous Boltzmann equation without angular cutoff, *Kinet. Relat. Models* 1 (2008), no. 3, 453-489.
- [14] X. Lu and C. Mouhot, On Measure Solutions of the Boltzmann Equation, part I: Moment Production and Stability Estimates, *J. Differential Equations* 4 (2012), 3305-3363.
- [15] S. Mischler and C. Mouhot, Kacs program in kinetic theory, *Inventiones mathematicae* (1) 193 (2013), 1-47.
- [16] S. Mischler, C. Mouhot, and B. Wennberg, A new approach to quantitative propagation of chaos for drift, diffusion and jump processes, *Probability Theory and Related Fields* 161.1-2 (2015), 1-59.
- [17] S. Mischler and B. Wennberg, On the spatially homogeneous Boltzmann equation, *Ann. Inst. H. Poincaré Anal. Non Linéaire* 16 (1999), 467-501.

- [18] C. Mouhot and C. Villani, Regularity theory for the spatially homogeneous Boltzmann equation with cut-off, *Arch. Ration. Mech. Anal.* 173 (2004), 169-212.
- [19] L. Silvestre, A new regularization mechanism for the Boltzmann equation without cut-off, *Commun. Math. Phys.* 348 (2016), 69-100.

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